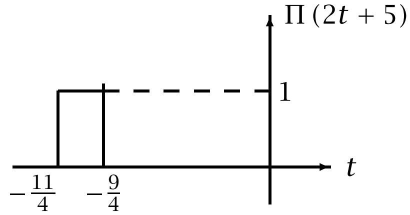
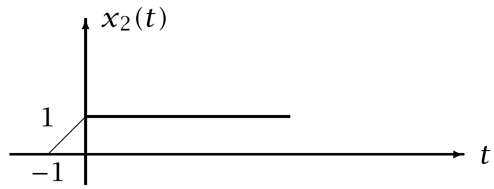
Chapter 2

Problem 2.1

1. $\Pi(2t+5) = \Pi\left(2\left(t+\frac{5}{2}\right)\right)$. This indicates first we have to plot $\Pi(2t)$ and then shift it to left by $\frac{5}{2}$. A plot is shown below:

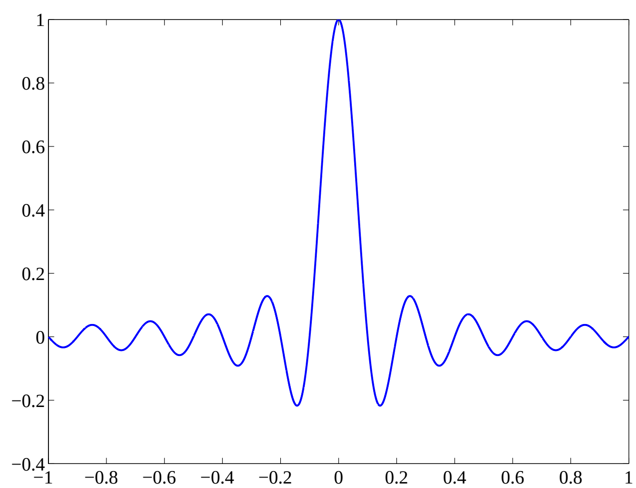


2. $\sum_{n=0}^{\infty} \Lambda(t-n)$ is a sum of shifted triangular pulses. Note that the sum of the left and right side of triangular pulses that are displaced by one unit of time is equal to 1, The plot is given below



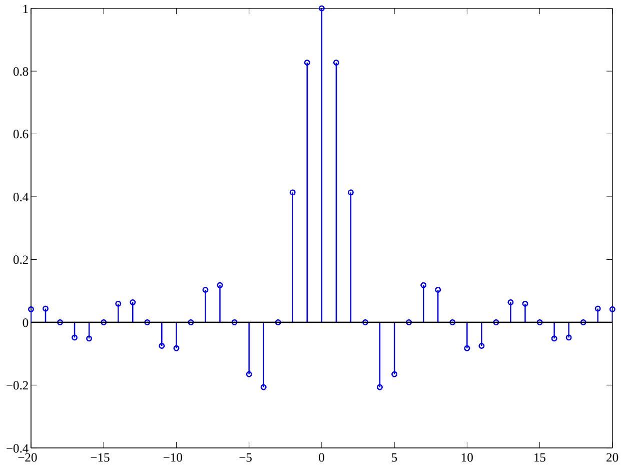
3. It is obvious from the definition of $\text{sgn}(t)$ that $\text{sgn}(2t) = \text{sgn}(t)$. Therefore $x_3(t) = 0$.

4. $x_4(t)$ is $\text{sinc}(t)$ contracted by a factor of 10.

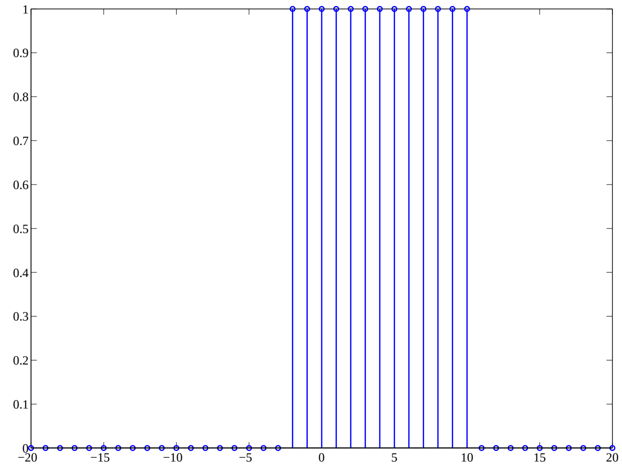


Problem 2.2

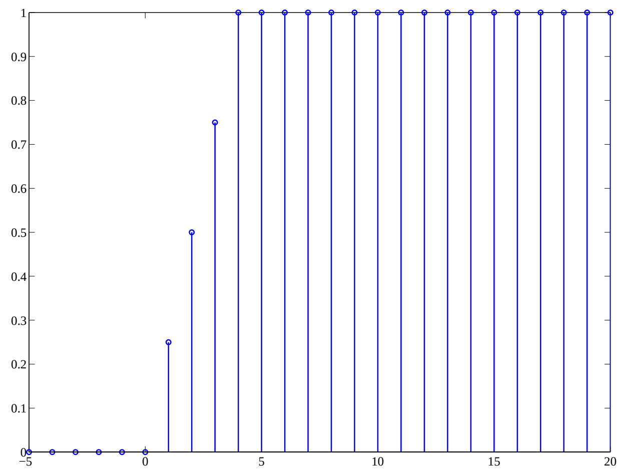
1. $x[n] = \text{sinc}(3n/9) = \text{sinc}(n/3)$.



2. $x[n] = \Pi\left(\frac{\frac{n}{4}-1}{3}\right)$. If $-\frac{1}{2} \leq \frac{\frac{n}{4}-1}{3} \leq \frac{1}{2}$, i.e., $-2 \leq n \leq 10$, we have $x[n] = 1$.



3. $x[n] = \frac{n}{4}u_{-1}(n/4) - (\frac{n}{4} - 1)u_{-1}(n/4 - 1)$. For $n < 0$, $x[n] = 0$, for $0 \leq n \leq 3$, $x[n] = \frac{n}{4}$ and for $n \geq 4$, $x[n] = \frac{n}{4} - \frac{n}{4} + 1 = 1$.



Problem 2.3

$x_1[n] = 1$ and $x_2[n] = \cos(2\pi n) = 1$, for all n. This shows that two signals can be different but their sampled versions be the same.

Problem 2.4

Let $x_1[n]$ and $x_2[n]$ be two periodic signals with periods N_1 and N_2, respectively, and let $N = \text{LCM}(N_1, N_2)$, and define $x[n] = x_1[n] + x_2[n]$. Then obviously $x_1[n+N] = x_1[n]$ and $x_2[n+N] = x_2[n]$, and hence $x[n] = x[n+N]$, i.e., $x[n]$ is periodic with period N.

For continuous-time signals $x_1(t)$ and $x_2(t)$ with periods T_1 and T_2 respectively, in general we cannot find a T such that $T = k_1 T_1 = k_2 T_2$ for integers k_1 and k_2. This is obvious for instance if $T_1 = 1$ and $T_2 = \pi$. The necessary and sufficient condition for the sum to be periodic is that $\frac{T_1}{T_2}$ be a rational number.

Problem 2.5

Using the result of problem 2.4 we have:

1. The frequencies are 2000 and 5500, their ratio (and therefore the ratio of the periods) is rational, hence the sum is periodic.

2. The frequencies are 2000 and $\frac{5500}{\pi}$. Their ratio is not rational, hence the sum is not periodic.

3. The sum of two periodic discrete-time signal is periodic.

4. The fist signal is periodic but $\cos[11000n]$ is *not* periodic, since there is no N such that $\cos[11000(n+N)] = \cos(11000n)$ for all n. Therefore the sum cannot be periodic.

Problem 2.6

1)

$$x_1(t) = \begin{cases} e^{-t} & t > 0 \\ -e^{t} & t < 0 \\ 0 & t = 0 \end{cases} \Longrightarrow x_1(-t) = \begin{cases} -e^{-t} & t > 0 \\ e^{t} & t < 0 \\ 0 & t = 0 \end{cases} = -x_1(t)$$

Thus, $x_1(t)$ is an odd signal

2) $x_2(t) = \cos\left(120\pi t + \frac{\pi}{3}\right)$ is neither even nor odd. We have $\cos\left(120\pi t + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t) - \sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$. Therefore $x_{2e}(t) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t)$ and $x_{2o}(t) = -\sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$. (Note: This part can also be considered as a special case of part 7 of this problem)

3)

$$x_3(t) = e^{-|t|} \Longrightarrow x_3(-t) = e^{-|(-t)|} = e^{-|t|} = x_3(t)$$

Hence, the signal $x_3(t)$ is even.

4)

$$x_4(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \Longrightarrow x_4(-t) = \begin{cases} 0 & t \geq 0 \\ -t & t < 0 \end{cases}$$

The signal $x_4(t)$ is neither even nor odd. The even part of the signal is

$$x_{4,e}(t) = \frac{x_4(t) + x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{-t}{2} & t < 0 \end{cases} = \frac{|t|}{2}$$

The odd part is

$$x_{4,o}(t) = \frac{x_4(t) - x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{t}{2} & t < 0 \end{cases} = \frac{t}{2}$$

5)

$$x_5(t) = x_1(t) - x_2(t) \Longrightarrow x_5(-t) = x_1(-t) - x_2(-t) = x_1(t) + x_2(t)$$

Clearly $x_5(-t) \neq x_5(t)$ since otherwise $x_2(t) = 0\ \forall t$. Similarly $x_5(-t) \neq -x_5(t)$ since otherwise $x_1(t) = 0\ \forall t$. The even and the odd parts of $x_5(t)$ are given by

$$\begin{aligned} x_{5,e}(t) &= \frac{x_5(t) + x_5(-t)}{2} = x_1(t) \\ x_{5,o}(t) &= \frac{x_5(t) - x_5(-t)}{2} = -x_2(t) \end{aligned}$$

Problem 2.7
For the first two questions we will need the integral $I = \int e^{ax}\cos^2 x dx$.

$$\begin{aligned}
I &= \frac{1}{a}\int \cos^2 x\, de^{ax} = \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a}\int e^{ax}\sin 2x\, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}\int \sin 2x\, de^{ax} \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\cos 2x\, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}(2\cos^2 x - 1)\, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\, dx - \frac{4}{a^2}I
\end{aligned}$$

Thus,

$$I = \frac{1}{4+a^2}\left[(a\cos^2 x + \sin 2x) + \frac{2}{a}\right]e^{ax}$$

1)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_1^2(t)dx = \lim_{T\to\infty}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_0^{\frac{T}{2}} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2\frac{T}{2} + \sin T - 1)e^{-T} + 3\right] = \frac{3}{8}
\end{aligned}$$

Thus $x_1(t)$ is an energy-type signal and the energy content is $3/8$

2)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\left[\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt + \int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt\right]
\end{aligned}$$

But,

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt &= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_{-\frac{T}{2}}^{0} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[-3 + (2\cos^2\frac{T}{2} + 1 + \sin T)e^{T}\right] = \infty
\end{aligned}$$

since $2 + \cos\theta + \sin\theta > 0$. Thus, $E_x = \infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt + \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$ is zero and

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt &= \lim_{T\to\infty} \frac{1}{8T}\left[2\cos^2\frac{T}{2} + 1 + \sin T\right]e^T \\
&> \lim_{T\to\infty} \frac{1}{T}e^T > \lim_{T\to\infty} \frac{1}{T}(1+T+T^2) > \lim_{T\to\infty} T = \infty
\end{aligned}$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_3^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \operatorname{sgn}^2(t)dt = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} T = \infty \\
P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \operatorname{sgn}^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{1}{T}T = 1
\end{aligned}$$

The signal $x_3(t)$ is of the power-type and the power content is 1.

4)

First note that

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi f t)dt = \sum_{k=-\infty}^{\infty} A\int_{k-\frac{1}{2f}}^{k+\frac{1}{2f}} \cos(2\pi f t)dt = 0$$

so that

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f t)dt &= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2 + A^2\cos(2\pi 2 f t))dt \\
&= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{2}A^2 T = \infty
\end{aligned}$$

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\
&= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_1 t)dt + \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} B^2\cos^2(2\pi f_2 t)dt + \\
&\quad AB\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\cos^2(2\pi(f_1+f_2) + \cos^2(2\pi(f_1-f_2)]dt \\
&= \infty + \infty + 0 = \infty
\end{aligned}$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type we consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$\begin{aligned}
P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A+B)^2\cos^2(2\pi f_1)dt \\
&= \lim_{T\to\infty}\frac{1}{2T}(A+B)^2\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \frac{1}{2}(A+B)^2
\end{aligned}$$

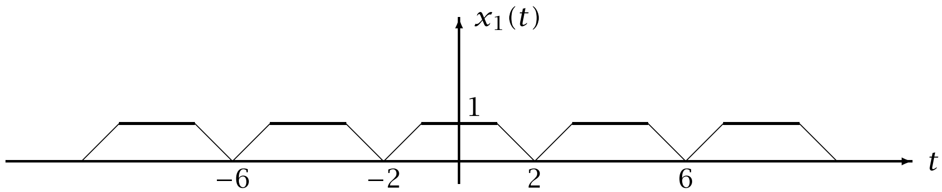
If $f_1 \neq f_2$ then

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2 \cos^2(2\pi f_1 t) + B^2 \cos^2(2\pi f_2 t) + 2AB \cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\ &= \lim_{T\to\infty} \frac{1}{T}\left[\frac{A^2 T}{2} + \frac{B^2 T}{2}\right] = \frac{A^2}{2} + \frac{B^2}{2} \end{aligned}$$

Thus the signal is of the power-type and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$ the power content is $\frac{1}{2}(A^2 + B^2)$

Problem 2.8

1. Let $x(t) = 2\Lambda\left(\frac{t}{2}\right) - \Lambda(t)$, then $x_1(t) = \sum_{n=-\infty}^{\infty} x(t-4n)$. First we plot $x(t)$ then by shifting it by multiples of 4 we can plot $x_1(t)$. $x(t)$ is a triangular pulse of width 4 and height 2 from which a standard triangular pulse of width 1 and height 1 is subtracted. The result is a trapezoidal pulse, which when replicated at intervals of 4 gives the plot of $x_1(t)$.



2. This is the sum of two periodic signals with periods 2π and 1. Since the ratio of the two periods is not rational the sum is not periodic (by the result of problem 2.4)

3. $\sin[n]$ is not periodic. There is no integer N such that $\sin[n+N] = \sin[n]$ for all n.

Problem 2.9

1)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} A^2 \left|e^{j(2\pi f_0 t + \theta)}\right|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty} \frac{1}{T} A^2 T = A^2$$

Thus $x(t) = Ae^{j(2\pi f_0 t + \theta)}$ is a power-type signal and its power content is A^2.

2)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} A^2 \cos^2(2\pi f_0 t + \theta)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \frac{A^2}{2} dt + \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \frac{A^2}{2} \cos(4\pi f_0 t + 2\theta)\, dt$$

As $T \to \infty$, the there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is $\frac{A^2}{2}$.

3)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} u_{-1}^2(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Thus the unit step signal is a power-type signal and its power content is 1/2

4)

$$\begin{aligned} E_x &= \lim_{T\to\infty} \int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty} \int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}} dt = \lim_{T\to\infty} 2K^2 t^{\frac{1}{2}} \Big|_0^{T/2} \\ &= \lim_{T\to\infty} \sqrt{2} K^2 T^{\frac{1}{2}} = \infty \end{aligned}$$

Thus the signal is not an energy-type signal.

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}} dt \\ &= \lim_{T\to\infty} \frac{1}{T} 2K^2 t^{\frac{1}{2}} \Big|_0^{T/2} = \lim_{T\to\infty} \frac{1}{T} 2K^2 (T/2)^{\frac{1}{2}} = \lim_{T\to\infty} \sqrt{2} K^2 T^{-\frac{1}{2}} = 0 \end{aligned}$$

Since P_x is not bounded away from zero it follows by definition that the signal is not of the power-type (recall that power-type signals should satisfy $0 < P_x < \infty$).

Problem 2.10

$$\Lambda(t) = \begin{cases} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{o.w.} \end{cases} \qquad u_{-1}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

Thus, the signal $x(t) = \Lambda(t)u_{-1}(t)$ is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ -t+1 & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases} \implies x(-t) = \begin{cases} 0 & t \le -1 \\ t+1 & -1 \le t < 0 \\ 1/2 & t = 0 \\ 0 & t > 0 \end{cases}$$

The even and the odd part of $x(t)$ are given by

$$\begin{aligned} x_e(t) &= \frac{x(t)+x(-t)}{2} = \frac{1}{2}\Lambda(t) \\ x_o(t) &= \frac{x(t)-x(-t)}{2} = \begin{cases} 0 & t \le -1 \\ \frac{-t-1}{2} & -1 \le t < 0 \\ 0 & t = 0 \\ \frac{-t+1}{2} & 0 < t \le 1 \\ 0 & 1 \le t \end{cases} \end{aligned}$$

Problem 2.11

1) Suppose that

$$x(t) = x_e^1(t) + x_o^1(t) = x_e^2(t) + x_o^2(t)$$

with $x_e^1(t)$, $x_e^2(t)$ even signals and $x_o^1(t)$, $x_o^1(t)$ odd signals. Then, $x(-t) = x_e^1(t) - x_o^1(t)$ so that

$$\begin{aligned} x_e^1(t) &= \frac{x(t) + x(-t)}{2} \\ &= \frac{x_e^2(t) + x_o^2(t) + x_e^2(-t) + x_o^2(-t)}{2} \\ &= \frac{2x_e^2(t) + x_o^2(t) - x_o^2(t)}{2} = x_e^2(t) \end{aligned}$$

Thus $x_e^1(t) = x_e^2(t)$ and $x_o^1(t) = x(t) - x_e^1(t) = x(t) - x_e^2(t) = x_o^2(t)$

2) Let $x_e^1(t)$, $x_e^2(t)$ be two even signals and $x_o^1(t)$, $x_o^2(t)$ be two odd signals. Then,

$$\begin{aligned} y(t) = x_e^1(t)x_e^2(t) &\implies y(-t) = x_e^1(-t)x_e^2(-t) = x_e^1(t)x_e^2(t) = y(t) \\ z(t) = x_o^1(t)x_o^2(t) &\implies z(-t) = x_o^1(-t)x_o^2(-t) = (-x_o^1(t))(-x_o^2(t)) = z(t) \end{aligned}$$

Thus the product of two even or odd signals is an even signal. For $v(t) = x_e^1(t)x_o^1(t)$ we have

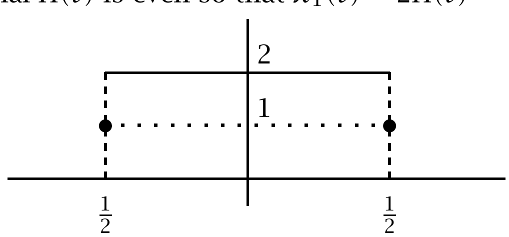
$$v(-t) = x_e^1(-t)x_o^1(-t) = x_e^1(t)(-x_o^1(t)) = -x_e^1(t)x_o^1(t) = -v(t)$$

Thus the product of an even and an odd signal is an odd signal.

3) One trivial example is $t + 1$ and $\frac{t^2}{t+1}$.

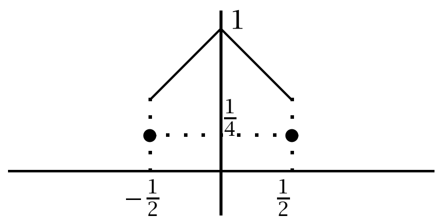
Problem 2.12

1) $x_1(t) = \Pi(t) + \Pi(-t)$. The signal $\Pi(t)$ is even so that $x_1(t) = 2\Pi(t)$

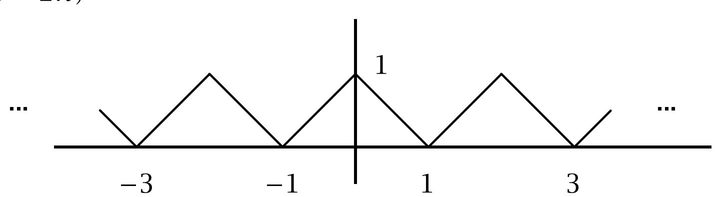


2)

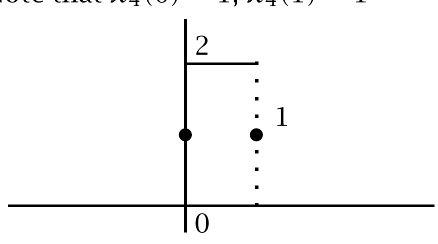
$$x_2(t) = \Lambda(t) \cdot \Pi(t) = \begin{cases} 0, & t < -1/2 \\ 1/4, & t = -1/2 \\ t+1, & -1/2 < t \le 0 \\ -t+1, & 0 \le t < 1/2 \\ 1/4, & t = 1/2 \\ 0, & 1/2 < t \end{cases}$$



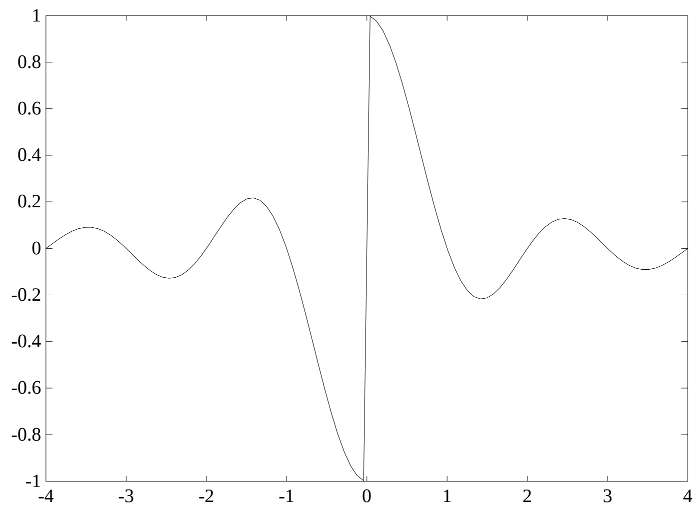
3) $x_3(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$



4) $x_4(t) = \mathrm{sgn}(t) + \mathrm{sgn}(1-t)$. Note that $x_4(0) = 1$, $x_4(1) = 1$



5) $x_5(t) = \mathrm{sinc}(t)\mathrm{sgn}(t)$. Note that $x_5(0) = 0$.



Problem 2.13
1) The value of the expression $\mathrm{sinc}(t)\delta(t)$ can be found by examining its effect on a function $\phi(t)$ through the integral

$$\int_{-\infty}^{\infty} \phi(t)\mathrm{sinc}(t)\delta(t) = \phi(0)\mathrm{sinc}(0) = \mathrm{sinc}(0)\int_{-\infty}^{\infty} \phi(t)\delta(t)$$

Thus $\mathrm{sinc}(t)\delta(t)$ has the same effect as the function $\mathrm{sinc}(0)\delta(t)$ and we conclude that

$$x_1(t) = \mathrm{sinc}(t)\delta(t) = \mathrm{sinc}(0)\delta(t) = \delta(t)$$

2) $\mathrm{sinc}(t)\delta(t-3) = \mathrm{sinc}(3)\delta(t-3) = 0$.

3)

$$\begin{aligned}
x_3(t) &= \Lambda(t) \star \sum_{n=-\infty}^{\infty} \delta(t-2n) \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(t-\tau)\delta(\tau-2n)d\tau \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(\tau-t)\delta(\tau-2n)d\tau \\
&= \sum_{n=-\infty}^{\infty} \Lambda(t-2n)
\end{aligned}$$