Chapter 1 Solutions

An Introduction to Mathematical Thinking: Algebra and Number Systems

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Exercise 1-1:

Determine which of the following sentences are statements. What are the truth values of those that are statements?
$7 > 5$

Solution:
It is a statement and it is true.

Exercise 1-2:

Determine which of the following sentences are statements. What are the truth values of those that are statements?
$5 > 7$

Solution:
It is a statement and its truth value is FALSE.

Exercise 1-3:

Determine which of the following sentences are statements. What are the truth values of those that are statements?
Is $5 > 7$?

Solution:
It is not a statement because it is a question.

Exercise 1-4:

Determine which of the following sentences are statements. What are the truth values of those that are statements?
$\sqrt{2}$ is an integer.

Solution:
This is a statement. It is false as there is no integer whose square is 2.

Exercise 1-5:

Determine which of the following sentences are statements. What are the truth values of those that are statements?

Show that $\sqrt{2}$ is not an integer.

Solution:

It is not a statement because the sentence does not have a truth value, it is a command.

Exercise 1-6:

Determine which of the following sentences are statements. What are the truth values of those that are statements?
If 5 is even then $6 = 7$.

Solution:

It is a statement and its truth value is TRUE.

Exercise 1-7:

Write down the truth tables for each expression. NOT(NOT P).

Solution:

P	NOT P	NOT(NOT P)
T	F	T
F	T	F

Exercise 1-8:

Write down the truth tables for each expression. NOT(P OR Q)

Solution:

P	Q	P OR Q	NOT (P OR Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Exercise 1-9:

Write down the truth tables for each expression. $P \Longrightarrow (Q \text{ OR } R)$

Solution:

P	Q	R	Q OR R	$P \Longrightarrow (Q \text{ OR } R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Exercise 1-10:

Write down the truth tables for each expression. $(P \text{ AND } Q) \implies R$

Solution:

P	Q	R	P AND Q	$(P \text{ AND } Q) \implies R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Exercise 1-11:

Write down the truth tables for each expression. $(P \text{ OR NOT } Q) \implies R$.

Solution:

P	Q	R	NOT Q	P OR (NOT Q)	$(P \text{ OR NOT } Q) \implies R$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	F

Exercise 1-12:

Write down the truth tables for each expression. NOT $P \implies (Q \iff R)$.

Solution:

P	Q	R	$Q \iff R$	NOT $P \implies (Q \iff R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	T	T

Exercise 1-13:

P UNLESS Q is defined as (NOT Q) $\Longrightarrow P$. Show that this statement has the same truth table as P OR Q. Give an example in common English showing the equivalence of P UNLESS Q and P OR Q.

Solution:

Defining P UNLESS Q as (NOT Q) $\Longrightarrow P$, then

P	Q	NOT Q	P UNLESS Q	P OR Q
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	F

Since the last two columns are the same the statement P UNLESS Q defined as (NOT Q) $\Longrightarrow P$ is equivalent to P OR Q.

"I will go unless I forget" and "I will go or I forget".

Exercise 1-14:

Write down the truth table for the *exclusive or* connective XOR, where the statement P XOR Q means (P OR Q) AND NOT (P AND Q). Show that this is equivalent to NOT($P \Longleftrightarrow Q$).

Solution:

P	Q	P OR Q	P AND Q	P XOR Q	NOT($P \Longleftrightarrow Q$)
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	F	F

Since the last two columns are the same, the statements are equivalent.

Exercise 1-15:

Write down the truth table for the *not or* connective NOR, where the statement P NOR Q means NOT(P OR Q).

Solution:

Defining P NOR Q as NOT(P OR Q), then the truth table for the *NOR* connective is

P	Q	P OR Q	P NOR Q
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Exercise 1-16:

Write down the truth table for the *not and* connective NAND, where the statement P NAND Q means NOT(P AND Q).

Solution:

P	Q	P AND Q	P NAND Q
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Exercise 1-17: Write each statement using P, Q, and connectives.
P whenever Q.

Solution:
$Q \Longrightarrow P$.

Exercise 1-18: Write each statement using P, Q, and connectives.
P is necessary for Q

Solution:
$Q \Longrightarrow P$.

Exercise 1-19: Write each statement using P, Q, and connectives.
P is sufficient for Q.

Solution:
$P \Longrightarrow Q$.

Exercise 1-20: Write each statement using P, Q, and connectives.
P only if Q

Solution:
$P \Longrightarrow Q$.

Exercise 1-21: Write each statement using P, Q, and connectives.
P is necessary and sufficient for Q.

Solution:
$P \Longleftrightarrow Q$. Another equivalent answer is $Q \Longleftrightarrow P$.

Exercise 1-22:

Show that the statements NOT (P OR Q) and (NOT P) AND (NOT Q) have the same truth tables and give an example of the equivalence of these statements in everyday language.

Solution:

P	Q	P OR Q	NOT (P OR Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

P	Q	NOT P	NOT Q	(NOT P) AND (NOT Q)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The final columns of each table are the same, so the two statements have the same truth tables.

This equivalence can be illustrated in everyday language. Consider the statement "I do not want cabbage or broccoli". This means that "I do not want cabbage" and "I do not want broccoli".

Exercise 1-23:

Show that the statements P AND (Q AND R) and (P AND Q) AND R have the same truth tables. This is the *associative law* for AND.

Solution:

P	Q	R	Q AND R	P AND (Q AND R)
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

P	Q	R	P AND Q	(P AND Q) AND R
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

The final columns of each table are equal, so the two statements have the same truth tables.

Exercise 1-24:

Show that the statements P AND (Q OR R) and (P AND Q) OR (P AND R) have the same truth tables. This is a *distributive law*.

Solution:

P	Q	R	Q OR R	P AND (Q OR R)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

P	Q	R	P AND Q	P AND R	(P AND Q) OR (P AND R)
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

The final columns of each table are the same, so the two statements have the same truth tables.

Exercise 1-25:

Is (P AND Q) $\Longrightarrow R$ equivalent to $P \Longrightarrow (Q \Longrightarrow R)$? Give reasons.

Solution 1:

Suppose (P AND Q) $\Longrightarrow R$ is false. Then P AND Q is true and R is false. Because both P and Q are true then $Q \Longrightarrow R$ is false, and thus $P \Longrightarrow (Q \Longrightarrow R)$ is also false.

Now suppose that $P \Longrightarrow (Q \Longrightarrow R)$ is false. Then P is true and $(Q \Longrightarrow R)$ is false. This last statement implies that Q is true and R is false. Therefore P AND Q is true, and (P AND Q) $\Longrightarrow R$ is false.

We have shown that whenever one statement is false, then the other one is also false. It follows that the statements are equivalent.

Solution 2:

P	Q	R	P AND Q	(P AND Q) $\Longrightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

P	Q	R	$Q \Longrightarrow R$	$P \Longrightarrow (Q \Longrightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

The final columns of each table are the same, so the two statements have the same truth tables, and the statements are equivalent.

Exercise 1-26:

Let P be the statement 'It is snowing' and let Q be the statement 'It is freezing.' Write each statement using P, Q, and connectives.
It is snowing, then it is freezing

Solution:
$P \Longrightarrow Q$.

Exercise 1-27:

Let P be the statement 'It is snowing' and let Q be the statement 'It is freezing.' Write each statement using P, Q, and connectives.
It is freezing but not snowing,

Solution:
Q AND (NOT P).

Exercise 1-28:

Let P be the statement 'It is snowing' and let Q be the statement 'It is freezing.' Write each statement using P, Q, and connectives.
When it is not freezing, it is not snowing.

Solution:
NOT $Q \Longrightarrow$ NOT P.

Exercise 1-29:

Let P be the statement 'I can walk,' Q be the statement 'I have broken my leg' and R be the statement 'I take the bus.' Express each statement as an English sentence.
$Q \implies$ NOT P.

Solution:
It can be "If I have broken my leg then I cannot walk".

Exercise 1-30:

Let P be the statement 'I can walk,' Q be the statement 'I have broken my leg' and R be the statement 'I take the bus.' Express each statement as an English sentence.
$P \iff$ NOT Q

Solution:
It can be "I can walk if and only if I have not broken my leg".

Exercise 1-31:

Let P be the statement 'I can walk' Q be the statement 'I have broken my leg' and R be the statement 'I take the bus.' Express each statement as an English sentence.
$R \implies (Q$ OR NOT $P)$

Solution:
It can be "If I take the bus then I have broken my leg or I cannot walk".

Exercise 1-32:

Let P be the statement 'I can walk,' Q be the statement 'I have broken my leg' and R be the statement 'I take the bus.' Express each statement as an English sentence.
$R \implies (Q \iff$ NOT $P)$

Solution:
It can be "I take the bus only if I have broken my leg is equivalent to I cannot walk".

Exercise 1-33:

Express each statement as a logical expression using quantifiers. State the universe of discourse.
There is a smallest positive integer.

Solution:
If we assume that the universe of discourse is the set of integers, we can express the statement as

$$\exists x \forall y,\ (0 < x \leq y).$$

Exercise 1-34:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

There is no smallest positive real number.

Solution:

The universe of discourse is the set of all positive real numbers. The statement "there is no smallest positive real number" is equivalent to

$$\forall r \exists x, \quad (x < r).$$

Exercise 1-35:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

Every integer is the product of two integers.

Solution:

If we assume that the universe of discourse is the set of integers, we can express the statement as

$$\forall x \exists y \exists z \ (x = yz).$$

Exercise 1-36:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

Every pair of integers has a common divisor.

Solution:

The universe of discourse is the set of integers. The given statement is

$$\forall x \forall y \exists z, \ (z \text{ divides } x \ \text{ AND } \ z \text{ divides } y).$$

Exercise 1-37:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

There is a real number x such that, for every real number y, $x^3 + x = y$.

Solution:

If we assume that the universe of discourse is the set of real numbers, we can express the statement as

$$\exists x \forall y,\ (x^3 + x = y).$$

Exercise 1-38:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

For every real number y, there is a real number x such that $x^3 + x = y$.

Solution:

If we assume that the universe of discourse is the set of real numbers, we can express the statement as

$$\forall y \exists x,\ (x^3 + x = y).$$

Exercise 1-39:

Express each statement as a logical expression using quantifiers. State the universe of discourse.

The equation $x^2 - 2y^2 = 3$ has an integer solution.

Solution:

If we assume that the universe of discourse is the set of integers, we can express the statement as

$$\exists x \exists y,\ (x^2 - 2y^2 = 3).$$

Exercise 1-40:

Express the following quote due to Abraham Lincoln as a logical expression using quantifiers: "You can fool some of the people all of the time, and all of the people some of the time, but you can not fool all of the people all of the time."

Solution:

Let x and t be variables. Let the universe of discourse of x to be the set of all people, and the universe of discourse of t to be the set of all times. And Let $F(x, t)$ stand for fooling a person x at time t.

The quote from Abraham Lincoln can be expressed as

$$\exists x \forall t,\ F(x, t) \text{ AND } \forall x \exists t,\ F(x, t) \text{ AND NOT } (\forall x \forall t,\ F(x, t)).$$

Exercise 1-41: Negate each expression, and simplify your answer.
$\forall x,\ (P(x) \text{ OR } Q(x))$

Solution:

$$\begin{aligned}
&\text{NOT } [\forall x,\ (P(x) \text{ OR } Q(x))] \\
&\exists x,\ \text{NOT } (P(x) \text{ OR } Q(x)) \\
&\exists x,\ (\text{NOT } P(x) \ \text{ AND } \ \text{NOT } Q(x)).
\end{aligned}$$

Exercise 1-42: Negate each expression, and simplify your answer.
$\forall x,\ ((P(x) \text{ AND } Q(x)) \Longrightarrow R(x))$.

Solution:
Using **Example 1.23.**, NOT $(A \Longrightarrow B)$ is equivalent to A AND NOT B, we have

$$\begin{aligned}
&\text{NOT } \forall x,\ [(P(x) \text{ AND } Q(x)) \Longrightarrow R(x)] \\
&\exists x,\ \text{NOT } [(P(x) \text{ AND } Q(x)) \Longrightarrow R(x)] \\
&\exists x,\ [(P(x) \text{ AND } Q(x)) \text{ AND NOT } R(x)]
\end{aligned}$$

Exercise 1-43: Negate each expression, and simplify your answer.
$\exists x,\ (P(x) \Longrightarrow Q(x))$.

Solution:
Using **Example 1.23.**, NOT $(A \Longrightarrow B)$ is equivalent to A AND NOT B, we have

$$\begin{aligned}
&\text{NOT } \exists x\ (P(x) \Longrightarrow Q(x)) \\
&\forall x,\ \text{NOT } (P(x) \Longrightarrow Q(x)) \\
&\forall x,\ (P(x) \ \text{ AND } \ \text{NOT } Q(x)).
\end{aligned}$$

Exercise 1-44: Negate each expression, and simplify your answer.
$\exists x\ \forall y,\ (P(x) \text{ AND } Q(y))$.

Solution:

$$\begin{aligned}
&\text{NOT } \exists x\ \forall y,\ (P(x) \text{ AND } Q(y))] \\
&\forall x \text{ NOT } \forall y,\ (P(x) \text{ AND } Q(y)) \\
&\forall x\ \exists y,\ \text{NOT } (P(x) \text{ AND } Q(y)) \\
&\forall x\ \exists y,\ (\text{NOT } P(x)) \text{ OR } (\text{NOT } Q(y))
\end{aligned}$$

Exercise 1-45:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\forall x\ \forall y,\ (x \geq y)$.

Solution:
Every real number is as large as any real number. This statement is false, if you let $x = 1$ and $y = 2$ then $1 < 2$.

Exercise 1-46:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\exists x \, \exists y, \quad (x \geq y)$.

Solution:

For some real number there is a real number that is less than or equal to it. This statement is always true because we can always take $y = x/2$

Exercise 1-47:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\exists y \, \forall x, \ (x \geq y)$.

Solution:

There is a smallest real number. This statement is false, if y is the smallest real number and you let $x = y - 1$ then $y - 1 < y$.

Exercise 1-48:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\forall x \, \exists y, \ (x \geq y)$.

Solution:

For every real number there is a smaller or equal real number.

This statement is true, if you let $y = x/2$ then $x \geq x/2$.

Exercise 1-49:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\forall x \, \exists y, \ (x^2 + y^2 = 1)$.

Solution:

For all real numbers x there exists a real number y such that $x^2 + y^2 = 1$.

This statement is false, if you let $|x| > 1$ then $1 - x^2 < 0$ and for all real y, $y^2 \geq 0$, so there is no real number y satisfying the equation.

Exercise 1-50:

If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?
$\exists y \, \forall x, \ (x^2 + y^2 = 1)$.

Solution:

There exists a real number y such that for all real numbers x, $x^2 + y^2 = 1$.

This statement is false, for every y let $|x| > 1$ then $x^2 > 1$ and because $y^2 > 0$ then $x^2 + y^2 > 1$.

Exercise 1-51:

Determine whether each pair of statements is equivalent. Give reasons.
$\exists x,\ (P(x)$ OR $Q(x))$. $\quad$ $(\exists x,\ P(x))$ OR $(\exists x,\ Q(x))$.

Solution:

These statements are equivalent. Suppose $\exists x,\ (P(x)$ OR $Q(x))$ is true. Hence $\exists x,,\ P(x)$ is true or $Q(x)$ is true. We can assume that there exists an x such that $P(x)$ is true, therefore for that particular x, $(\exists x, P(x))$ OR $(\exists x,\ Q(x))$ is true regardless of the value of $\exists x,\ Q(x)$. This also holds if $\exists x,\ Q(x)$ is true.

Now suppose that $(\exists x,\ P(x))$ OR $(\exists x,\ Q(x))$ is true. Hence at least one of $(\exists x\ P(x))$ or $(\exists x\ Q(x))$ is true. Assume that there exists an x such that $P(x)$ is true, therefore for that particular x, $P(x)$ OR $Q(x)$ is true regardless of the value of $Q(x)$. So $\exists x,\ (P(x)$ OR $Q(x))$ is true. This also holds if $(\exists x,\ Q(x))$ is true.

We have shown that whenever one of the statements is true, then the other one is also true. Hence they are equivalent.

Exercise 1-52:

Determine whether each pair of statements is equivalent. Give reasons.
$\exists x,\ (P(x)$ AND $Q(x))$. $\quad$ $(\exists x\ P(x))$ AND $(\exists x,\ Q(x))$.

Solution:

These statements are not equivalent. Assume the universe of discourse is the set of real numbers. Let $P(x)$ be the statement $x > 0$ and $Q(x)$ the statement $x \leq 0$. Then $\exists x,\ (P(x)$ AND $Q(x))$ is false while $(\exists x,\ P(x))$ AND $(\exists x,\ Q(x))$ is true. (It may not be the same x in both parts of the second statement!)

Exercise 1-53:

Determine whether each pair of statements is equivalent. Give reasons.
$\forall x,\ (P(x) \Longrightarrow Q(x))$. $\quad$ $(\forall x,\ P(x)) \Longrightarrow (\forall x,\ Q(x))$.

Solution:

These statements are not always equivalent. We can give a particular example in which they do not have the same meaning.

Let the universe of discourse be the set of real numbers. Let $P(x)$ be the expression $x < 0$ and $Q(x)$ be the expression $x^2 < 0$. Then for all real numbers x, if $x < 0$ then $x^2 > 0$ so $P(x) \Longrightarrow Q(x)$ is false. However, $(\forall x,\ P(x))$ is not true, and $(\forall x,\ Q(x))$ is not true so $(\forall x,\ P(x)) \Longrightarrow (\forall x,\ Q(x))$ is true.

Exercise 1-54:

Determine whether each pair of statements is equivalent. Give reasons.
$\forall x,\ (P(x)$ OR $Q(y))$. $\quad$ $(\forall x,\ P(x))$ OR $Q(y)$.

Solution:

These statements are equivalent. Because the variable x does not occur in $Q(y)$, this statement does not depend on the quantifiers of x, it depends only on the particular choice of y.

Therefore, the statement $\forall x,\ (P(x) \text{ OR } Q(y))$ is true when $\forall x,\ P(x)$ is true or when $Q(y)$ is true. This is exactly the second statement.

Exercise 1-55: Write the contrapositive, and the converse of each statement.
If Tom goes to the party then I will go to the party.

Solution:
Contrapositive: If I don't go to the party the Tom will not go to the party.
Converse: If I go to the party then Tom will go to the party.

Exercise 1-56: Write the contrapositive, and the converse of each statement.
If I do my assignments then I get a good mark in the course.

Solution:
Contrapositive: If I do not get a good mark in the course then I do not do my assignments.
Converse: If I get a good mark in the course then I do my assignments.

Exercise 1-57: Write the contrapositive, and the converse of each statement.
If $x > 3$ then $x^2 > 9$.

Solution:
Contrapositive: If $x^2 \leq 9$ then $x \leq 3$.
Converse: If $x^2 > 9$ then $x > 3$.

Exercise 1-58: Write the contrapositive, and the converse of each statement.
If $x < -3$ then $x^2 > 9$.

Solution:
Contrapositive: If $x^2 \leq 9$ then $x \geq -3$.
Converse: If $x^2 > 9$ then $x < -3$.

Exercise 1-59: Write the contrapositive, and the converse of each statement.
If an integer is divisible by 2 then it is not prime.

Solution:
Contrapositive: If an integer is a prime then it is not divisible by 2.
Converse: If an integer is not prime then it is divisible by 2.

Exercise 1-60: Write the contrapositive, and the converse of each statement.
If $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

Solution:
Contrapositive: If $xy < 0$ then $x < 0$ or $y < 0$.
Converse: If $xy \geq 0$ then $x \geq 0$ and $y \geq 0$.

Exercise 1-61: Write the contrapositive, and the converse of each statement.
If $x^2 + y^2 = 9$ then $-3 \leq x \leq 3$.

Solution:

Contrapositive: If $x < -3$ OR $x > 3$ then $x^2 + y^2 \neq 9$.
Converse: If $-3 \leq x \leq 3$ then $x^2 + y^2 = 9$.

Exercise 1-62:

Let S and T be sets. Prove that if $x \notin S \cap T$ then $x \notin S$ or $x \notin T$.

Solution:

We can proceed by proving the contrapositive of the statement. That is if $x \in S$ and $x \in T$ then $x \in S \cap T$.

If $x \in S$ AND $x \in T$ then by definition of intersection of sets $x \in S \cap T$.

By the Contrapositive Law we have proved the original statement.

Exercise 1-63:

Let a and b be real numbers. Prove that if $ab = 0$ then $a = 0$ or $b = 0$.

Solution:

Suppose that a and b are real numbers such that $ab = 0$ and that $a \neq 0$. Therefore $1/a$ exists. Multiplying both sides of the equation by it gives

$$\begin{aligned} ab &= 0 \\ \tfrac{1}{a}ab &= 0 \cdot \tfrac{1}{a} \\ b &= 0. \end{aligned}$$

So we have shown

$$((a, b \in \mathbb{R}, ab = 0) \text{ AND NOT } (a = 0)) \Longrightarrow (b = 0).$$

This is equivalent to the original statement

$$(a, b \in \mathbb{R}, ab = 0) \Longrightarrow (a = 0 \text{ OR } b = 0).$$

Exercise 1-64: Use the Contrapositive Proof Method to prove that

$$(S \cap T = \emptyset) \text{ AND } (S \cup T = T) \Longrightarrow S = \emptyset.$$

Solution:

We want to prove the contrapositive of the statement. That is

$$(S \neq \emptyset) \Longrightarrow (S \cap T \neq \emptyset) \text{ OR } (S \cup T \neq T).$$

Because $S \neq \emptyset$ then $\exists x, (x \in S)$. Assume also that $S \cap T = \emptyset$. It follows that $x \notin T$, and because $x \in S \cup T$ then $S \cup T \neq T$.

We have shown

$$(S \neq \emptyset) \text{ AND NOT } (S \cap T \neq \emptyset) \Longrightarrow (S \cup T \neq T).$$

This is equivalent to the statement

$$(S \neq \emptyset) \Longrightarrow (S \cap T \neq \emptyset) \text{ OR } (S \cup T \neq T).$$

Thus by the Contrapositive Law we have proven the original statement.

Exercise 1-65: Prove or give a counterexample to each statement.
$\forall x \in \mathbb{R}\ (x^2 + 5x + 7 > 0)$.

Solution:
We will prove the statement by direct proof. By completing the square we get

$$x^2 + 5x + 7 = \left(x + \frac{5}{2}\right)^2 + \frac{3}{4}.$$

$(x + 5/2)^2 \geq 0$ for all $x \in \mathbb{R}$ so

$$x^2 + 5x + 7 = \left(x + \frac{5}{2}\right)^2 + \frac{3}{4} > 0.$$

Therefore the result is true.

Exercise 1-66: Prove or give a counterexample to each statement.
If m and n are integers with mn odd, then m and n are odd.

Solution:
Using Proof Method 1.58 we shall split the proof into two cases one for m and the other for n. Suppose that m is even then $m = 2k$ for some integer k. Therefore $mn = 2kn$. Because kn is also an integer then mn must be even. By the Contrapositive Law we have proved that if mn is odd then m is odd. By the symmetry of m and n, it follows that if mn is odd then n is also odd.

Hence if m and n are integers with mn odd, then both m and n are odd.

Exercise 1-67: Prove or give a counterexample to each statement.
If x and y are real numbers then $\forall x\ \exists y\ (x^2 > y^2)$.

Solution:
The statement is not true. As an easy counter example let $x = 0$, then for every $y \in \mathbb{R}$, $y^2 \geq 0 = x^2$.

Exercise 1-68: Prove or give a counterexample to each statement.
$(S \cap T) \cup U = S \cap (T \cup U)$, for any sets S, T, and U.

Solution:
The statement is false. To see this notice that for any set A, $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

Let $S = \emptyset$, T any set and $U \neq \emptyset$. Then $(S \cap T) \cup U = \emptyset \cup U = U$, but $S \cap (T \cup U) = \emptyset$. And by our assumptions $U \neq \emptyset$.

Exercise 1-69: Prove or give a counterexample to each statement.
$S \cup T = T \iff S \subseteq T$.

Solution:
We shall prove the statement.
We will first prove $S \cup T = T \Longrightarrow S \subseteq T$ by direct proof. If $x \in S$ then $x \in S \cup T$. Since $S \cup T = T$ then $x \in T$. This proves that $S \subseteq T$, as desired.
To prove the other direction, $S \subseteq T \Longrightarrow S \cup T = T$, let $x \in S \cup T$. Hence $x \in S$ or $x \in T$ (or both). If $x \in S$ then, since $S \subseteq T$, $x \in T$. Hence x is always in T. This proves that $S \cup T \subseteq T$. Because it is always true that $T \subseteq S \cup T$, we can conclude that $S \cup T = T$.

Exercise 1-70: Prove or give a counterexample to each statement.
If x is a real number such that $x^4 + 2x^2 - 2x < 0$ then $0 < x < 1$.

Solution:
We shall prove the statement.
Using Proof Method 1.58, we will split the proof into two cases,

$$x^4 + 2x^2 - 2x < 0 \Longrightarrow 0 < x \quad \text{and} \quad x^4 + 2x^2 - 2x < 0 \Longrightarrow x < 1.$$

If $x \leq 0$ then $x^4 + 2x^2 - 2x \geq 0$, since each term is nonnegative. By the Contrapositive Proof Method this proves the first case.
Now, if $x \geq 1$ then $x^4 \geq 1$ and $2x(x-1) \geq 1$, so $x^4 + 2x^2 - 2x \geq 1 \geq 0$. By the Contrapositive Proof Method this proves the second case.
Hence if x is a real number such that $x^4 + 2x^2 - 2x < 0$ then $0 < x < 1$.

Exercise 1-71: Prove the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution:
If $x \in A \cap (B \cup C)$ then $x \in A$ AND $x \in B \cup C$. And $x \in B \cup C$ implies $x \in B$ OR $x \in C$. If $x \notin A \cap B$ then $x \in C$ and so $x \in A \cap C$. So $x \in (A \cap B) \cup (A \cap C)$ and

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Also if $x \in (A \cap B) \cup (A \cap C)$ then $x \in (A \cap B)$ OR $x \in (A \cap C)$. If $x \in (A \cap B)$ then $x \in A$ and $x \in (B \cup C)$. This is also true if $x \in (A \cap C)$. Therefore $x \in A \cap (B \cup C)$ and

$$A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C).$$

It follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1-72: Prove the distributive law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution:
If $x \in A \cup (B \cap C)$ then $x \in A$ OR $x \in B \cap C$. If $x \in A$ then $x \in (A \cup B)$ AND $x \in (A \cup C)$. This is also true if $x \in B \cap C$, since $x \in B$ AND $x \in C$. Therefore $x \in (A \cup B) \cap (A \cup C)$ and

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C).$$

Also if $x \in (A \cup B) \cap (A \cup C)$ then $x \in (A \cup B)$ AND $x \in (A \cup C)$. If x is in both $(A \cup B)$ and $(A \cup C)$, but $x \notin A$ then $x \in B$ AND $x \in C$ so $x \in B \cap C$. Therefore $x \in A \cup (B \cap C)$ and

$$A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C).$$

It follows that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 1-73:

If S, T and U are sets, the statement $S \cap T \subseteq U$ can be expressed as

$$\forall x \ \ ((x \in S \text{ AND } x \in T) \Longrightarrow x \in U).$$

Express and simplify the negation of this expression, namely $S \cap T \not\subseteq U$, in terms of quantifiers.

Solution:

We negate the expression using **Example 1.23.**, NOT $(A \Longrightarrow B)$ is equivalent to A AND NOT B, to get

$$\begin{array}{l} \text{NOT } \forall x \ \ ((x \in S \text{ AND } x \in T) \Longrightarrow x \in U) \\ \exists x \ \ \text{NOT } ((x \in S \text{ AND } x \in T) \Longrightarrow x \in U) \\ \exists x \ ((x \in S \text{ AND } x \in T) \text{ AND } \ \text{NOT } (x \in U)) \\ \exists x \ ((x \in S) \text{ AND } (x \in T) \text{ AND } (x \notin U)) \end{array}$$

Problem 1-74:

If S and T are sets, the statement $S = T$ can be expressed as

$$\forall x \ \ (x \in S \Longleftrightarrow x \in T).$$

What does $S \neq T$ mean? How would you go about showing that two sets are not the same?

Solution:

We use the fact that $A \Longleftrightarrow B$ is equivalent to $(A \Longrightarrow B)$ AND $(B \Longrightarrow A)$ and by Example 1.23, that NOT $(A \Longrightarrow B)$ is equivalent to A AND NOT B.

We negate the expression $S = T$ to get

$$\begin{array}{l} \text{NOT } \forall x \ \ (x \in S \Longleftrightarrow x \in T) \\ \exists x \ \ \text{NOT } (x \in S \Longrightarrow x \in T \text{ AND } x \in T \Longrightarrow x \in S) \\ \exists x \ \ \text{NOT } (x \in S \Longrightarrow x \in T) \text{ OR } \ \text{NOT } (x \in T \Longrightarrow x \in S) \\ \exists x \ (x \in S \text{ AND } x \notin T) \text{ OR } (x \in T \text{ AND } x \notin S). \end{array}$$

So, you can show that two sets S and T are not the same either by finding an element $x \in S$ but $x \notin T$, or by finding an element $y \in T$ but $y \notin S$.

Problem 1-75:

The definition of the limit of a function, $\lim_{x \to a} f(x) = L$, can be expressed using quantifiers as

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon).$$

Use quantifiers to express the negation of this statement, which would be a definition of $\lim_{x \to a} f(x) \neq L$.

Solution:

We negate the definition of the limit of a function to get

$$\begin{array}{l}
\text{NOT } \ \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon) \\
\exists \epsilon > 0 \ \ \text{NOT } \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon) \\
\exists \epsilon > 0 \ \forall \delta > 0 \ \ \text{NOT } \forall x \ (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon) \\
\exists \epsilon > 0 \ \forall \delta > 0 \ \exists x \ \text{ NOT } (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon) \\
\exists \epsilon > 0 \ \forall \delta > 0 \ \exists x \ (0 < |x - a| < \delta \text{ AND } \ \text{NOT } |f(x) - L| < \epsilon) \\
\exists \epsilon > 0 \ \forall \delta > 0 \ \exists x \ (0 < |x - a| < \delta \text{ AND } |f(x) - L| \geq \epsilon)
\end{array}$$

This is the definition of $\lim_{x \to a} f(x) \neq L$.

Problem 1-76:

Use truth tables to show that the statement $P \Longrightarrow (Q \text{ OR } R)$ is equivalent to the statement $(P \text{ AND NOT } Q) \Longrightarrow R$.
[This explains the Proof Method 1.56 for $P \Longrightarrow (Q \text{ OR } R)$.]

Solution:

P	Q	R	Q OR R	$P \Longrightarrow (Q \text{ OR } R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

P	Q	R	NOT Q	P AND NOT Q	$(P \text{ AND NOT } Q) \Longrightarrow R$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Because the two final columns are the same, the two statements have the same truth value therefore they are equivalent.

Problem 1-77:

Use truth tables to show that the statement $(P \text{ OR } Q) \Longrightarrow R$ is equivalent to the statement $(P \Longrightarrow R)$ AND $(Q \Longrightarrow R)$.
[This explains the Proof Method 1.57 for $(P \text{ OR } Q) \Longrightarrow R$.]

Solution:

P	Q	R	P OR Q	$(P \text{ OR } Q) \Longrightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

P	Q	R	$P \Longrightarrow R$	$Q \Longrightarrow R$	$(P \Longrightarrow R)$ AND $(Q \Longrightarrow R)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Because the two final columns are the same, the two statements have the same truth value therefore they are equivalent.

Problem 1-78:

Use truth tables to show that the statement $P \Longrightarrow (Q \text{ AND } R)$ is equivalent to the statement $(P \Longrightarrow Q)$ AND $(P \Longrightarrow R)$.
[This explains the Proof Method 1.58 for $P \Longrightarrow (Q \text{ AND } R)$.]

Solution:

P	Q	R	Q AND R	$P \Longrightarrow (Q \text{ AND } R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

P	Q	R	$P \Longrightarrow Q$	$P \Longrightarrow R$	$(P \Longrightarrow Q)$ AND $(P \Longrightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Because the two final columns are the same, the two statements have the same truth value therefore they are equivalent.

Problem 1-79:

Is the statement $(P$ AND $Q) \Longrightarrow R$ equivalent to $(P \Longrightarrow R)$ OR $(Q \Longrightarrow R)$? Give reasons.

Solution 1:

The statements are equivalent. Suppose that $(P$ AND $Q) \Longrightarrow R$ is true. Hence $(P$ AND $Q)$ is false or R is true. If $(P$ AND $Q)$ is false then P or Q is false, so at least one of $(P \Longrightarrow Q)$ or $(P \Longrightarrow R)$ is true. On the other hand if R is true then both $(P \Longrightarrow R)$, $(P \Longrightarrow Q)$ are true. In both cases $(P \Longrightarrow R)$ OR $(Q \Longrightarrow R)$ is true.

Now suppose that $(P \Longrightarrow R)$ OR $(Q \Longrightarrow R)$ is true. Hence at least one of $(P \Longrightarrow R)$ and $(Q \Longrightarrow R)$ is true. Without loss of generality, we can assume that $(P \Longrightarrow R)$ is true. Therefore P is false or R is true, so $(P$ AND $Q)$ is false or R is true. In both cases $(P$ AND $Q) \Longrightarrow R$ is true.

We have shown that whenever one of the statements is true, then the other one is also true. Hence they are equivalent.

Solution 2:

P	Q	R	P AND Q	$(P$ AND $Q) \Longrightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

P	Q	R	$P \implies R$	$Q \implies R$	$(P \implies R)$ OR $(Q \implies R)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Because the two final columns are the same, the two statements have the same truth value therefore they are equivalent.

Problem 1-80:

Is the statement $P \implies (Q \implies R)$ equivalent to $(P \implies Q) \implies R$? Give reasons.

Solution 1:

The statements are not equivalent. To see this let P, Q and R be all false. Then $(Q \implies R)$ and $(P \implies Q)$ are true. Therefore $P \implies (Q \implies R)$ is true but $(P \implies Q) \implies R$ is false.

Solution 2:

P	Q	R	$Q \implies R$	$P \implies (Q \implies R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

P	Q	R	$P \implies Q$	$(P \implies Q) \implies R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

Since the final columns are not the same, the two statements are not equivalent. In particular, they differ in the bottom row, so the truth value of the statements are different when P, Q and R are all false.

Problem 1-81:

Show that the statement P OR Q OR R is equivalent to the statement

$$(\text{ NOT } P \text{ AND } \text{ NOT } Q) \Longrightarrow R.$$

Solution 1:

To avoid ambiguity, we first have to show that the statements derived from reading P OR Q OR R from left to right and from right to left are equivalent. That is

$$P \text{ OR } (Q \text{ OR } R) \textit{ is equivalent to } (P \text{ OR } Q) \text{ OR } R.$$

Now P OR (Q OR R) is false when P and (Q OR R) are both false. And (Q OR R) is false when Q and R are both false. And when P, Q and R are all false so is (P OR Q) OR R).

Now, (P OR Q) OR R is false when (P OR Q) and R are both false. And (P OR Q) is false when P and Q are both false. And when P, Q and R are all false so is P OR (Q OR R).

We have shown that whenever one statement is false, then the other one is also false, therefore they are equivalent.

Using,

$$\begin{array}{rll} \text{NOT } (A \text{ AND } B) & \textit{is equivalent to} & (\text{NOT } A) \text{ OR } (\text{NOT } B) \\ \text{NOT } (A \text{ OR } B) & \textit{is equivalent to} & (\text{NOT } A) \text{ AND } (\text{NOT } B) \\ \text{NOT } (A \Longrightarrow B) & \textit{is equivalent to} & A \text{ AND } (\text{ NOT } B) \end{array}$$

then,

$$\begin{array}{l} P \text{ OR } Q \text{ OR } R \\ \text{NOT NOT } [\, (P \text{ OR } Q) \text{ OR } R \,] \\ \text{NOT } [\, (\text{NOT } P \text{ AND NOT } Q) \text{ AND NOT } R \,] \\ \text{NOT NOT } [\, (\text{ NOT } P \text{ AND NOT } Q) \Longrightarrow R \,] \\ (\text{ NOT } P \text{ AND NOT } Q) \Longrightarrow R \end{array}$$

Solution 2:

The statement P OR Q OR R is true if at least one of P, Q, and R is true; that is, it is true unless P, Q, and R are all false.

P	Q	R	P OR Q OR R
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

P	Q	R	NOT P AND NOT Q	(NOT P AND NOT Q) $\Longrightarrow R$)
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

Because the two final columns are the same, the two statements have the same truth value therefore they are equivalent.

Problem 1-82:

For each truth table, find a statement involving P and Q and the connectives, AND, OR, and NOT, that yields that truth table.

P	Q	???
T	T	T
T	F	T
F	T	F
F	F	T

Solution:

The truth table is very similar to the truth table of $P \Longrightarrow Q$, except that the two rows in the middle yield opposite values. This indicates that the truth table corresponds to the statement $Q \Longrightarrow P$. Using the connectives AND, OR and NOT we have

$$Q \Longrightarrow P$$
$$\text{NOT [NOT } (Q \Longrightarrow P) \text{]}$$
$$\text{NOT [} Q \text{ AND NOT } P \text{]}$$
$$\text{NOT } Q \text{ OR } P$$
$$P \text{ OR NOT } Q.$$

Therefore P OR NOT Q yields the given truth table.

Check:

P	Q	NOT Q	P OR NOT Q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Problem 1-83:

For each truth table, find a statement involving P and Q and the connectives, AND, OR, and NOT, that yields that truth table.

P	Q	???
T	T	F
T	F	T
F	T	F
F	F	F

Solution:

The truth table is the negation of $P \implies Q$. And NOT $(P \implies Q)$ is equivalent to P AND NOT Q.

Check:

P	Q	NOT Q	P AND NOT Q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Problem 1-84:

(a) How many nonequivalent statements are there involving P and Q?

(b) How many nonequivalent statements are there involving $P_1, P_2, \ldots, P_n$?

Solution:

(a) Two statements involving P and Q are equivalent if they have the same truth tables. The number of nonequivalent statements is the number of truth different truth tables there are with P and Q. The truth tables with P and Q have four rows. Since each row has two possible values, T and F, the number of possibilities for the four rows is $2^4 = 16$. Hence, there are 16 nonequivalent statements involving P and Q.

[Note that these 16 nonequivalent statements include 4 that can be written without using both P and Q, namely: P, NOT P, Q, and NOT Q. However P, for example, could be written as P OR (Q AND NOT Q), since the expression in brackets is always false.]

(b) We can count the number of nonequivalent statements involving the statements $P_1, P_2, \ldots, P_n$ by counting the different truth tables there are with them. The number of rows in the truth table of a statement involving n unknowns is 2^n. Since each row has two possible values, T and F, the number of possibilities for the 2^n rows is 2^{2^n}. This is the number of nonequivalent statements involving $P_1, P_2, \ldots, P_n$.