☐ DIAGNOSTIC TESTS

Test A   Algebra

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(b) $-3^4 = -(3)(3)(3)(3) = -81$

(c) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

(d) $\dfrac{5^{23}}{5^{21}} = 5^{23-21} = 5^2 = 25$

(e) $\left(\dfrac{2}{3}\right)^{-2} = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$

(f) $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{\left(\sqrt[4]{16}\,\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

2. (a) Note that $\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$ and $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. Thus $\sqrt{200} - \sqrt{32} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}$.

(b) $(3a^3b^3)(4ab^2)^2 = 3a^3b^3 16a^2b^4 = 48a^5b^7$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \left(\dfrac{x^2y^{-1/2}}{3x^{3/2}y^3}\right)^2 = \dfrac{(x^2y^{-1/2})^2}{(3x^{3/2}y^3)^2} = \dfrac{x^4y^{-1}}{9x^3y^6} = \dfrac{x^4}{9x^3y^6y} = \dfrac{x}{9y^7}$

3. (a) $3(x+6) + 4(2x-5) = 3x + 18 + 8x - 20 = 11x - 2$

(b) $(x+3)(4x-5) = 4x^2 - 5x + 12x - 15 = 4x^2 + 7x - 15$

(c) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{b} - \left(\sqrt{b}\right)^2 = a - b$

Or: Use the formula for the difference of two squares to see that $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b.$

(d) $(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9.$

Note: A quicker way to expand this binomial is to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$:

$(2x+3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$

(e) See Reference Page 1 for the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using it, we get

$(x+2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8.$

4. (a) Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we have

$4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5).$

(b) Factoring by trial and error, we get $2x^2 + 5x - 12 = (2x-3)(x+4).$

(c) Using factoring by grouping and the difference of two squares formula, we have

$x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x^2-4)(x-3) = (x-2)(x+2)(x-3).$

(d) $x^4 + 27x = x(x^3 + 27) = x(x+3)(x^2 - 3x + 9)$

This last expression was obtained using the sum of two cubes formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$. [See Reference Page 1 in the textbook.]

(e) The smallest exponent on x is $-\frac{1}{2}$, so we will factor out $x^{-1/2}$.

$3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x-1)(x-2)$

(f) $x^3y - 4xy = xy(x^2 - 4) = xy(x-2)(x+2)$

5. (a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2} = \dfrac{(x+1)(x+2)}{(x+1)(x-2)} = \dfrac{x+2}{x-2}$

(b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x+3}{2x+1} = \dfrac{(2x+1)(x-1)}{(x-3)(x+3)} \cdot \dfrac{x+3}{2x+1} = \dfrac{x-1}{x-3}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} \cdot \dfrac{x-2}{x-2} = \dfrac{x^2 - (x+1)(x-2)}{(x-2)(x+2)}$

$= \dfrac{x^2 - (x^2 - x - 2)}{(x+2)(x-2)} = \dfrac{x+2}{(x+2)(x-2)} = \dfrac{1}{x-2}$

(d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}} = \dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}} \cdot \dfrac{xy}{xy} = \dfrac{y^2 - x^2}{x - y} = \dfrac{(y-x)(y+x)}{-(y-x)} = \dfrac{y+x}{-1} = -(x+y)$

6. (a) $\dfrac{\sqrt{10}}{\sqrt{5} - 2} = \dfrac{\sqrt{10}}{\sqrt{5} - 2} \cdot \dfrac{\sqrt{5} + 2}{\sqrt{5} + 2} = \dfrac{\sqrt{50} + 2\sqrt{10}}{\left(\sqrt{5}\right)^2 - 2^2} = \dfrac{5\sqrt{2} + 2\sqrt{10}}{5 - 4} = 5\sqrt{2} + 2\sqrt{10}$

(b) $\dfrac{\sqrt{4+h} - 2}{h} = \dfrac{\sqrt{4+h} - 2}{h} \cdot \dfrac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \dfrac{4 + h - 4}{h\left(\sqrt{4+h} + 2\right)} = \dfrac{h}{h\left(\sqrt{4+h} + 2\right)} = \dfrac{1}{\sqrt{4+h} + 2}$

7. (a) $x^2 + x + 1 = \left(x^2 + x + \tfrac{1}{4}\right) + 1 - \tfrac{1}{4} = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$

(b) $2x^2 - 12x + 11 = 2(x^2 - 6x) + 11 = 2(x^2 - 6x + 9 - 9) + 11 = 2(x^2 - 6x + 9) - 18 + 11 = 2(x-3)^2 - 7$

8. (a) $x + 5 = 14 - \tfrac{1}{2}x \;\Leftrightarrow\; x + \tfrac{1}{2}x = 14 - 5 \;\Leftrightarrow\; \tfrac{3}{2}x = 9 \;\Leftrightarrow\; x = \tfrac{2}{3} \cdot 9 \;\Leftrightarrow\; x = 6$

(b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x} \;\Rightarrow\; 2x^2 = (2x-1)(x+1) \;\Leftrightarrow\; 2x^2 = 2x^2 + x - 1 \;\Leftrightarrow\; x = 1$

(c) $x^2 - x - 12 = 0 \;\Leftrightarrow\; (x+3)(x-4) = 0 \;\Leftrightarrow\; x + 3 = 0 \text{ or } x - 4 = 0 \;\Leftrightarrow\; x = -3 \text{ or } x = 4$

(d) By the quadratic formula, $2x^2 + 4x + 1 = 0 \;\Leftrightarrow\;$

$$x = \dfrac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \dfrac{-4 \pm \sqrt{8}}{4} = \dfrac{-4 \pm 2\sqrt{2}}{4} = \dfrac{2\left(-2 \pm \sqrt{2}\right)}{4} = \dfrac{-2 \pm \sqrt{2}}{2} = -1 \pm \tfrac{1}{2}\sqrt{2}.$$

(e) $x^4 - 3x^2 + 2 = 0 \;\Leftrightarrow\; (x^2 - 1)(x^2 - 2) = 0 \;\Leftrightarrow\; x^2 - 1 = 0 \text{ or } x^2 - 2 = 0 \;\Leftrightarrow\; x^2 = 1 \text{ or } x^2 = 2 \;\Leftrightarrow\;$

$x = \pm 1 \text{ or } x = \pm\sqrt{2}$

(f) $3|x - 4| = 10 \;\Leftrightarrow\; |x - 4| = \tfrac{10}{3} \;\Leftrightarrow\; x - 4 = -\tfrac{10}{3} \text{ or } x - 4 = \tfrac{10}{3} \;\Leftrightarrow\; x = \tfrac{2}{3} \text{ or } x = \tfrac{22}{3}$

(g) Multiplying through $2x(4-x)^{-1/2} - 3\sqrt{4-x} = 0$ by $(4-x)^{1/2}$ gives $2x - 3(4-x) = 0 \;\Leftrightarrow\;$

$2x - 12 + 3x = 0 \;\Leftrightarrow\; 5x - 12 = 0 \;\Leftrightarrow\; 5x = 12 \;\Leftrightarrow\; x = \tfrac{12}{5}.$

9. (a) $-4 < 5 - 3x \le 17 \;\Leftrightarrow\; -9 < -3x \le 12 \;\Leftrightarrow\; 3 > x \ge -4 \text{ or } -4 \le x < 3.$

In interval notation, the answer is $[-4, 3)$.

(b) $x^2 < 2x + 8 \;\Leftrightarrow\; x^2 - 2x - 8 < 0 \;\Leftrightarrow\; (x+2)(x-4) < 0.$ Now, $(x+2)(x-4)$ will change sign at the critical values $x = -2$ and $x = 4$. Thus the possible intervals of solution are $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$. By choosing a single test value from each interval, we see that $(-2, 4)$ is the only interval that satisfies the inequality.

(c) The inequality $x(x-1)(x+2) > 0$ has critical values of $-2, 0$, and 1. The corresponding possible intervals of solution are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$ and $(1, \infty)$. By choosing a single test value from each interval, we see that both intervals $(-2, 0)$ and $(1, \infty)$ satisfy the inequality. Thus, the solution is the union of these two intervals: $(-2, 0) \cup (1, \infty)$.

(d) $|x - 4| < 3$ $\Leftrightarrow$ $-3 < x - 4 < 3$ $\Leftrightarrow$ $1 < x < 7$. In interval notation, the answer is $(1, 7)$.

(e) $\dfrac{2x - 3}{x + 1} \le 1$ $\Leftrightarrow$ $\dfrac{2x - 3}{x + 1} - 1 \le 0$ $\Leftrightarrow$ $\dfrac{2x - 3}{x + 1} - \dfrac{x + 1}{x + 1} \le 0$ $\Leftrightarrow$ $\dfrac{2x - 3 - x - 1}{x + 1} \le 0$ $\Leftrightarrow$ $\dfrac{x - 4}{x + 1} \le 0$.

Now, the expression $\dfrac{x - 4}{x + 1}$ may change signs at the critical values $x = -1$ and $x = 4$, so the possible intervals of solution are $(-\infty, -1)$, $(-1, 4]$, and $[4, \infty)$. By choosing a single test value from each interval, we see that $(-1, 4]$ is the only interval that satisfies the inequality.

10. (a) False. In order for the statement to be true, it must hold for all real numbers, so, to show that the statement is false, pick $p = 1$ and $q = 2$ and observe that $(1 + 2)^2 \ne 1^2 + 2^2$. In general, $(p + q)^2 = p^2 + 2pq + q^2$.

(b) True as long as a and b are nonnegative real numbers. To see this, think in terms of the laws of exponents:

$$\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\,\sqrt{b}.$$

(c) False. To see this, let $p = 1$ and $q = 2$, then $\sqrt{1^2 + 2^2} \ne 1 + 2$.

(d) False. To see this, let $T = 1$ and $C = 2$, then $\dfrac{1 + 1(2)}{2} \ne 1 + 1$.

(e) False. To see this, let $x = 2$ and $y = 3$, then $\dfrac{1}{2 - 3} \ne \dfrac{1}{2} - \dfrac{1}{3}$.

(f) True since $\dfrac{1/x}{a/x - b/x} \cdot \dfrac{x}{x} = \dfrac{1}{a - b}$, as long as $x \ne 0$ and $a - b \ne 0$.

Test B   Analytic Geometry

1. (a) Using the point $(2, -5)$ and $m = -3$ in the point-slope equation of a line, $y - y_1 = m(x - x_1)$, we get
$$y - (-5) = -3(x - 2) \quad \Rightarrow \quad y + 5 = -3x + 6 \quad \Rightarrow \quad y = -3x + 1.$$

(b) A line parallel to the x-axis must be horizontal and thus have a slope of 0. Since the line passes through the point $(2, -5)$, the y-coordinate of every point on the line is -5, so the equation is $y = -5$.

(c) A line parallel to the y-axis is vertical with undefined slope. So the x-coordinate of every point on the line is 2 and so the equation is $x = 2$.

(d) Note that $2x - 4y = 3$ $\Rightarrow$ $-4y = -2x + 3$ $\Rightarrow$ $y = \frac{1}{2}x - \frac{3}{4}$. Thus the slope of the given line is $m = \frac{1}{2}$. Hence, the slope of the line we're looking for is also $\frac{1}{2}$ (since the line we're looking for is required to be parallel to the given line). So the equation of the line is $y - (-5) = \frac{1}{2}(x - 2)$ $\Rightarrow$ $y + 5 = \frac{1}{2}x - 1$ $\Rightarrow$ $y = \frac{1}{2}x - 6$.

2. First we'll find the distance between the two given points in order to obtain the radius, r, of the circle:
$r = \sqrt{[3 - (-1)]^2 + (-2 - 4)^2} = \sqrt{4^2 + (-6)^2} = \sqrt{52}$. Next use the standard equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center, to get $(x + 1)^2 + (y - 4)^2 = 52$.