1 THE GEOMETRY OF EUCLIDEAN SPACE

1.1: VECTORS IN TWO- AND THREE-DIMENSIONAL SPACE

GOALS

1. Be able to perform the following operations on vectors: addition, subtraction, scalar multiplication.

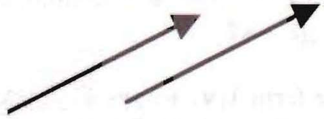
2. Given a vector and a point, be able to write the equation of the line passing through the point in the direction of the vector.

3. Given two points, be able to write the equation of the line passing through them.

STUDY HINTS

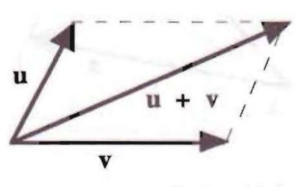
1. *Space notation.* The symbol $\mathbb{R}$ or $\mathbb{R}^1$ refers to all points on the real number line or a one-dimensional space. $\mathbb{R}^2$ refers to all ordered pairs (x, y) which lie in the plane, a two-dimensional space. $\mathbb{R}^3$ refers to all ordered triples (x, y, z) which lie in three-dimensional space. In general, the "exponent" in $\mathbb{R}^n$ tells you how many components there are in each vector.

2. *Vectors and scalars.* A vector has both length (magnitude) and direction. Scalars are just numbers. Scalars do not have direction. Two vectors are equal if and only if they both have the same length and the same direction. Pictorially, they do not need to originate from the same starting point. The vectors shown here are equal.



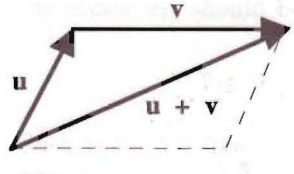
3. *Vector notation.* Vectors are often denoted by boldface letters, underlined letters, arrows over letters, or by an n-tuple $(x_1, x_2, ..., x_n)$. Each x_i of the n-tuple is called the i^{th} component. BEWARE that the n-tuple may represent either a point *or* a vector. The vector $(0, 0, ..., 0)$ is denoted **0**. Your instructor or other textbooks may use other notations such as a squiggly line underneath a letter. A circumflex over a letter is sometimes used to represent a *unit* vector.

4. *Vector addition.* Vectors may be added componentwise, *e.g.*, in $\mathbb{R}^2$

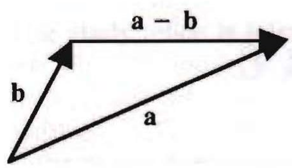


$$(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2).$$

Pictorially, two vectors may be thought of as the sides of a parallelogram. Starting from the vertex formed by the two vectors, we form a new vector which ends at the opposite corner of the parallelogram. This new vector is the sum of the other two. Alternatively, one could simply translate **v** so that the tail of **v** meets the head of **u**. The vector joining the tail of **u** to the head of **v** is **u** + **v**.



5. *Vector subtraction.* Just as with addition, vectors may be subtracted componentwise. Think of this as adding a negative vector. Pictorially, the vectors **a**, **b** and **a** − **b** form a triangle. To determine the correct direction, you should be able to add **a** − **b** and **b** to get **a**. Thus **a** − **b** goes from the tip of **b** to the tip of **a**.



6. *Scalar multiplication.* Here, each component of a vector is multiplied by the same scalar, *e.g.*, in $\mathbb{R}^2$,

$$r(x, y) = (rx, ry) \quad \text{for any real number } r.$$

The effect of multiplication by a positive scalar is to change the length by a factor. If the scalar is negative, the lengthening occurs in the opposite direction. Multiplication of vectors will be discussed in the next two sections.

7. *Standard basis vectors.* These are vectors whose components are all 0 except for a single 1. In $\mathbb{R}^3$, **i**, **j** and **k** denote the vectors which lie on the x, y and z axes. They are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. The standard basis vectors in $\mathbb{R}^2$ are **i** and **j**, which are vectors lying on the x and y axes, and their respective components are $(1, 0)$ and $(0, 1)$. Sometimes, these vectors are denoted by

$$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \quad \text{or} \quad \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3.$$

8. *Lines.* (a) The line passing through **a** in the direction of **v** is $\mathbf{l}(t) = a + t\mathbf{v}$. This is called the point-direction form of the line because the only necessary information is the point **a** and the direction of **v**.

   (b) The line passing through **a** and **b** is $\mathbf{l}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. This is called the point-point form of the line. To see if the direction is correct, plug in $t = 0$ and you should get the first point. Plug in $t = 1$ and you should get the second point.

9. *Spanning a space.* If all points in a space can be written in the form $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n$, where λ_i are scalars, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span that given space. For example, the vectors **i** and **j** span the xy plane.

10. *Geometric proofs.* The use of vectors can often simplify a proof. Try to compare vector methods and non-vector methods by doing example 10 without vectors.

SOLUTIONS TO SELECTED EXERCISES

1. We must solve the following equations:

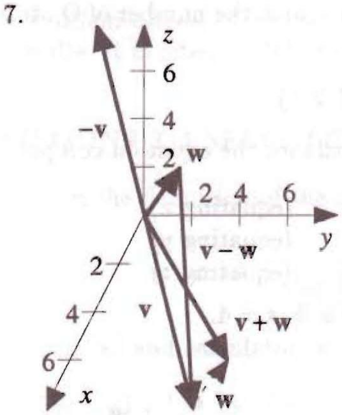
$$\begin{aligned} -21 - x &= -25 \\ 23 - 6 &= y. \end{aligned}$$

   We get $x = 4$ and $y = 17$, so $(-21, 23) - (4, 6) = (-25, 17)$.

4. Convert $-4\mathbf{i} + 3\mathbf{j}$ to $(-4, 3, 0)$, so

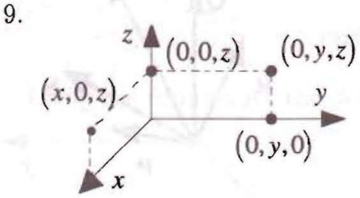
$$(2, 3, 5) - 4\mathbf{i} + 3\mathbf{j} = (2, 3, 5) + (-4, 3, 0) = (-2, 6, 5).$$

7.



To sketch $\mathbf{v}$, start at the origin and move 2 units along the x axis, then move 3 units parallel to the y axis, and then move -6 units parallel to the z axis. The vector $\mathbf{w}$ is sketched analogously. The vector $-\mathbf{v}$ has the same length as $\mathbf{v}$, but it points in the opposite direction. To sketch $\mathbf{v}+\mathbf{w}$, translate the tail of $\mathbf{w}$ to the head of $\mathbf{v}$ and draw the vector from the origin to the head of the translated $\mathbf{w}$. The vector $\mathbf{v}-\mathbf{w}$ goes from the head of $\mathbf{w}$ to the head of $\mathbf{v}$.

9.



On the y axis, points have the coordinates $(0,y,0)$, so we must restrict x and z to be 0. On the z axis, points have the coordinates $(0,0,z)$, so we must restrict x and y to be 0. In the xz plane, points have the coordinates $(x,0,z)$, so we must restrict y to be 0. In the yz plane, points have the coordinates $(0,y,z)$, so we must restrict x to be 0.

12. Every point on the plane spanned by the given vectors can be written as $a\mathbf{v}_1 + b\mathbf{v}_2$, where a and b are real numbers; therefore, the plane is described by

$$a(3,-1,1)+b(0,3,4).$$

15. Given two points $\mathbf{a}$ and $\mathbf{b}$, a line through them is $\mathbf{l}(t)=\mathbf{a}+t(\mathbf{b}-\mathbf{a})$. In this case, $\mathbf{a}=(-1,-1,-1)$ and $\mathbf{b}=(1,-1,2)$, so we get

$$\mathbf{l}(t)=(-1,-1,-1)+t(2,0,3)=(2t-1,-1,3t-1).$$

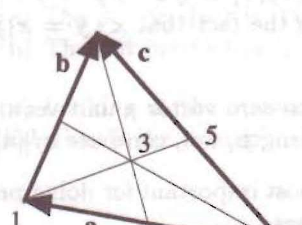
19. Substitute $\mathbf{v}=(x,y,z)=(2+t,-2+t,-1+t)$ into the equation for x, y and z and get

$$\begin{aligned} 2x-3y+z-2 &= 2(2+t)-3(-2+t)+(-1+t)-2 \\ &= 4+2t+6-3t-1+t-2=7. \end{aligned}$$

Since $7\neq 0$, there are no points (x,y,z) satisfying the equation and lying on $\mathbf{v}$.

23. Just as the parallelogram of example 17 was described by $\mathbf{v}=s\mathbf{a}+t\mathbf{b}$ for s and t in $[0,1]$, the parallelpiped can be described by $\mathbf{w}=s\mathbf{a}+t\mathbf{b}+r\mathbf{c}$, for s, t and r in $[0,1]$.

28.



Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the sides of the triangle as shown, and let $\mathbf{v}_{ij}$ denote the vector from point i to point j. We assume that each median is divided into a ratio of 2 : 1 by the point of intersection. Then we have

$$\begin{aligned} \mathbf{v}_{12} &= -\mathbf{a}/2=-(\mathbf{c}-\mathbf{b})/2; \\ \mathbf{v}_{23} &= (1/3)(\mathbf{a}/2+\mathbf{b}); \\ \mathbf{v}_{34} &= (-2/3)(\mathbf{a}+\mathbf{b}/2); \\ \mathbf{v}_{45} &= (\mathbf{a}+\mathbf{b})/2. \end{aligned}$$

The vector $\mathbf{v}_{15}$ should be the sum $\mathbf{v}_{12}+\mathbf{v}_{23}+\mathbf{v}_{34}+\mathbf{v}_{45}$, or

$$\mathbf{v}_{15}=-\frac{\mathbf{c}-\mathbf{b}}{2}+\frac{1}{3}\left(\frac{\mathbf{a}}{2}+\mathbf{b}\right)-\frac{2}{3}\left(\mathbf{a}+\frac{\mathbf{b}}{2}\right)+\frac{\mathbf{a}+\mathbf{b}}{2}=\mathbf{b}-\frac{\mathbf{c}}{2},$$

which is the median of the vector that ends on $\mathbf{c}$. The other two medians are analyzed the same way.