CHAPTER 1

Linear systems theory

Problems

Written exercises

1.1 Find the rank of the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution

The rank of a matrix A can be defined as the dimension of the largest submatrix consisting of rows and columns of A whose determinant is nonzero. With this definition we see that the rank of the zero matrix is zero.

1.2 Find two 2×2 matrices A and B such that $A \neq B$, neither A nor B are diagonal, $A \neq cB$ for any scalar c, and $AB = BA$. Find the eigenvectors of A and B. Note that they share an eigenvector. Interestingly, every pair of commuting matrices shares at least one eigenvector [Hor85, p. 51].

Optimal State Estimation Solution Manual, First Edition. By Dan J. Simon

Solution

Suppose A and B are given as

$$\begin{aligned} A &= \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \\ B &= \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \end{aligned}$$

Then we see that

$$\begin{aligned} AB &= \begin{bmatrix} a_1b_1 + a_2b_2 & a_1b_2 + a_2b_3 \\ a_2b_1 + a_3b_2 & a_2b_2 + a_3b_3 \end{bmatrix} \\ BA &= \begin{bmatrix} a_1b_1 + a_2b_2 & a_2b_1 + a_3b_2 \\ a_1b_2 + a_2b_3 & a_2b_2 + a_3b_3 \end{bmatrix} \end{aligned}$$

We see that $AB = BA$ if $a_1b_2 + a_2b_3 = a_2b_1 + a_3b_2$. This will be true, for example, if $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $b_1 = 1$, $b_2 = 3$, and $b_3 = 1$. This gives

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \end{aligned}$$

For these matrices A has the eigenvalues -1 and 3, B has the eigenvalues -2 and 4, and both A and B have the eigenvectors $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

1.3 Prove the three identities of Equation (1.26).

Solution

a). Suppose A is an $n \times m$ matrix, and B is an $m \times p$ matrix. Then

$$\begin{aligned} (AB)^T &= \left(\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{bmatrix} \right)^T \\ &= \begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{1j}B_{jp} \\ \vdots & \ddots & \vdots \\ \sum A_{nj}B_{j1} & \cdots & \sum A_{nj}B_{jp} \end{bmatrix}^T \\ &= \begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{nj}B_{j1} \\ \vdots & \ddots & \vdots \\ \sum A_{1j}B_{jp} & \cdots & \sum A_{nj}B_{jp} \end{bmatrix} \\ B^TA^T &= \begin{bmatrix} B_{11} & \cdots & B_{m1} \\ \vdots & \ddots & \vdots \\ B_{1p} & \cdots & B_{mp} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1m} & \cdots & A_{nm} \end{bmatrix} \\ &= \begin{bmatrix} \sum B_{j1}A_{1j} & \cdots & \sum B_{j1}A_{nj} \\ \vdots & \ddots & \vdots \\ \sum B_{jp}A_{1j} & \cdots & \sum B_{jp}A_{nj} \end{bmatrix} \end{aligned}$$

QED

b). Suppose that $(AB)^{-1} = C$. Then $CAB = I$. Postmultiplying both sides of this equation by B^{-1} gives $CA = B^{-1}$. Postmultiplying both sides of this equation by A^{-1} gives $C = B^{-1}A^{-1}$. Hence we see that $(AB)^{-1} = B^{-1}A^{-1}$. QED

c). Suppose A is an $n \times m$ matrix, and B is an $m \times n$ matrix. Then

$$
\begin{aligned}
\mathrm{Tr}(AB) &= \mathrm{Tr}\left(\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}\right) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{1j}B_{jn} \\ \vdots & \ddots & \vdots \\ \sum A_{nj}B_{j1} & \cdots & \sum A_{nj}B_{jn} \end{bmatrix}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\
\mathrm{Tr}(BA) &= \mathrm{Tr}\left(\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}\right) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \sum B_{1j}A_{j1} & \cdots & \sum B_{1j}A_{jm} \\ \vdots & \ddots & \vdots \\ \sum B_{mj}A_{j1} & \cdots & \sum B_{mj}A_{jm} \end{bmatrix}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij}A_{ji}
\end{aligned}
$$

QED

1.4 Find the partial derivative of the trace of AB with respect to A.

Solution

Suppose A is an $n \times m$ matrix, and B is an $m \times n$ matrix. Then

$$
\begin{aligned}
\frac{\partial \mathrm{Tr}(AB)}{\partial A} &= \frac{\partial}{\partial A}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\
&= \begin{bmatrix} \frac{\partial}{\partial A_{11}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} & \cdots & \frac{\partial}{\partial A_{1m}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial A_{n1}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} & \cdots & \frac{\partial}{\partial A_{nm}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \end{bmatrix} \\
&= \begin{bmatrix} B_{11} & \cdots & B_{m1} \\ \vdots & \ddots & \vdots \\ B_{1n} & \cdots & B_{mn} \end{bmatrix} \\
&= B^T
\end{aligned}
$$

1.5 Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Recall that the eigenvalues of A are found by find the roots of the polynomial $P(\lambda) = |\lambda I - A|$. Show that $P(A) = 0$. (This is an illustration of the Cayley–Hamilton theorem [Bay99, Che99, Kai00].)

Solution

$$\begin{aligned}
P(\lambda) &= |\lambda I - A| \\
&= \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} \\
&= \lambda^2 - (a+c)\lambda + ac - b^2 \\
P(A) &= A^2 - (a+c)A + (ac - b^2)I \\
&= \begin{bmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{bmatrix} - (a+c)\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} ac - b^2 & 0 \\ 0 & ac - b^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

1.6 Suppose that A is invertible and

$$\begin{bmatrix} A & A \\ B & A \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Find B and C in terms of A [Lie67].

Solution

Multiplying out the matrix equation gives the following two equations.

$$\begin{aligned}
A^2 + AC &= 0 \\
BA + AC &= I
\end{aligned}$$

Solving for B and C in terms of A gives

$$\begin{aligned}
B &= A + A^{-1} \\
C &= -A
\end{aligned}$$

1.7 Show that AB may not be symmetric even though both A and B are symmetric.

Solution

Suppose the symmetric matrices A and B are given as

$$\begin{aligned}
A &= \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \\
B &= \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix}
\end{aligned}$$