0.1 Basic Problems

0.1 Let $z = 8 + j3$ and $v = 9 - j2$,

(a) Find

$$(i)\ \mathcal{R}e(z) + \mathcal{I}m(v), \quad (ii)\ |z+v|, \quad (iii)\ |zv|, \quad (iv)\ \angle z + \angle v, \quad (v)\ |v/z|, \quad (vi)\ \angle(v/z)$$

(b) Find the trigonometric and polar forms of

$$(i)\ z+v, \quad (ii)\ zv, \quad (iii)\ z^* \quad (iv)\ zz^*, \quad (v)\ z-v$$

Answers: (a) $\mathcal{R}e(z) + \mathcal{I}m(v) = 6$; $|v/z| = \sqrt{85}/\sqrt{73}$; (b) $zz^* = |z|^2 = 73$.
Solution

(a)
i. $\mathcal{R}e(z) + \mathcal{I}m(v) = 8 - 2 = 6$

ii. $|z+v| = |17 + j1| = \sqrt{17^2 + 1}$

iii. $|zv| = |72 - j16 + j27 + 6| = |78 + j11| = \sqrt{78^2 + 11^2}$

iv. $\angle z + \angle v = \tan^{-1}(3/8) - \tan^{-1}(2/9)$

v. $|v/z| = |v|/|z| = \sqrt{85}/\sqrt{73}$

vi. $\angle(v/z) = -\tan^{-1}(2/9) - \tan^{-1}(3/8)$

(b)
i. $z + v = 17 + j = \sqrt{17^2 + 1}e^{j\tan^{-1}(1/17)}$
ii. $zv = 78 + j11 = \sqrt{78^2 + 11^2}e^{j\tan^{-1}(11/78)}$
iii. $z^* = 8 - j3 = \sqrt{64 + 9}(e^{-j\tan^{-1}(3/8)})^* = \sqrt{73}e^{j\tan^{-1}(3/8)}$
iv. $zz^* = |z|^2 = 73$
v. $z - v = -1 + j5 = \sqrt{1 + 25}e^{-j\tan^{-1}(5)}$

0.2 Use Euler's identity to

(a) show that

$$(i)\ \cos(\theta - \pi/2) = \sin(\theta), \ \ (ii)\ -\sin(\theta - \pi/2) = \cos(\theta), \ \ (iii)\ \cos(\theta) = \sin(\theta + \pi/2).$$

(b) to find

$$(i)\quad \int_0^1 \cos(2\pi t)\sin(2\pi t)dt, \ \ (ii)\quad \int_0^1 \cos^2(2\pi t)dt.$$

Answers: (b) 0 and $1/2$.

Solution

(a) We have

i. $\cos(\theta - \pi/2) = 0.5(e^{j(\theta-\pi/2)} + e^{-j(\theta-\pi/2)}) = -j0.5(e^{j\theta} - e^{-j\theta}) = \sin(\theta)$

ii. $-\sin(\theta - \pi/2) = 0.5j(e^{j(\theta-\pi/2)} - e^{-j(\theta-\pi/2)}) = 0.5j(-j)(e^{j\theta} + e^{-j\theta}) = \cos(\theta)$

iii. $\sin(\theta + \pi/2) = (je^{j\theta} + je^{-j\theta})/(2j) = \cos(\theta)$

(b) i. $\cos(2\pi t)\sin(2\pi t) = (1/4j)(e^{j4\pi t} - e^{-j4\pi t})$ so that

$$\int_0^1 \cos(2\pi t)\sin(2\pi t)dt = \frac{1}{4j}\frac{e^{j4\pi t}}{4\pi j}|_0^1 + \frac{1}{4j}\frac{e^{-j4\pi t}}{4\pi j}|_0^1 = 0 + 0 = 0$$

ii. We have

$$\cos^2(2\pi t) = \frac{1}{4}(e^{j4\pi t} + 2 + e^{-j4\pi t}) = \frac{1}{2}(1 + \cos(4\pi t))$$

so that its integral is $1/2$ since the integral of $\cos(4\pi t)$ is over two of its periods and it is zero.

0.3 Use Euler's identity to

(a) show the identities

$$(i) \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$(ii) \quad \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

(b) find an expression for $\cos(\alpha)\cos(\beta)$, and for $\sin(\alpha)\sin(\beta)$.

Answers: $e^{j\alpha}e^{j\beta} = \cos(\alpha+\beta) + j\sin(\alpha+\beta) = [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + j[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)]$.

Solution
(a) Using Euler's identity the product

$$\begin{aligned} e^{j\alpha}e^{j\beta} &= (\cos(\alpha) + j\sin(\alpha))(\cos(\beta) + j\sin(\beta)) \\ &= [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + j[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)] \end{aligned}$$

while

$$e^{j(\alpha+\beta)} = \cos(\alpha+\beta) + j\sin(\alpha+\beta)$$

so that equating the real and imaginary parts of the above two equations we get the desired trigonometric identities.

(b) We have

$$\begin{aligned} \cos(\alpha)\cos(\beta) &= 0.5(e^{j\alpha} + e^{-j\alpha})\,0.5(e^{j\beta} + e^{-j\beta}) \\ &= 0.25(e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}) + 0.25(e^{j(\alpha-\beta)} + e^{-j(\alpha-\beta)}) \\ &= 0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta) \end{aligned}$$

Now,

$$\begin{aligned} \sin(\alpha)\sin(\beta) &= \cos(\alpha - \pi/2)\cos(\beta - \pi/2) \\ &= 0.5\cos(\alpha - \pi/2 + \beta - \pi/2) + 0.5\cos(\alpha - \pi/2 - \beta + \pi/2) \\ &= 0.5\cos(\alpha + \beta - \pi) + 0.5\cos(\alpha - \beta) \\ &= -0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta) \end{aligned}$$

0.4 Consider the calculation of roots of an equation $z^N = \alpha$ where $N \geq 1$ is an integer and $\alpha = |\alpha|e^{j\phi}$ a nonzero complex number.

(a) First verify that there are exactly N roots for this equation and that they are given by $z_k = re^{j\theta_k}$ where $r = |\alpha|^{1/N}$ and $\theta_k = (\phi + 2\pi k)/N$ for $k = 0, 1, \cdots, N-1$.

(b) Use the above result to find the roots of the following equations

$$(i)\ z^2 = 1, \quad (ii)\ z^2 = -1, \quad (iii)\ z^3 = 1, \quad (iv)\ z^3 = -1.$$

and plot them in a polar plane (i..e., indicating their magnitude and phase). Explain how the roots are distributed in the polar plane.

Answers: Roots of $z^3 = -1 = 1e^{j\pi}$ are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$, equally spaced around circle of radius r.

Solution

(a) Replacing $z_k = |\alpha|^{1/N}e^{j(\phi+2\pi k)/N}$ in z^N we get $z_k^N = |\alpha|e^{j(\phi+2\pi k)} = |\alpha|e^{j(\phi)} = \alpha$ for any value of $k = 0, \cdots, N-1$.

(b) Applying the above result we have:

- For $z^2 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi} = -1$ and $z_1 = e^{j2\pi} = 1$.
- When $z^2 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi/2} = j$, and $z_1 = e^{j3\pi/2} = -j$.
- For $z^3 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j2\pi/3}$; for $k = 1$, $z_1 = e^{j4\pi/3} = e^{-j2\pi/3} = z_0^*$; and for $k = 2$, $z_2 = 1e^{j(2\pi)} = 1$.
- When $z^3 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j\pi/3}$; for $k = 1$, $z_1 = e^{j\pi} = -1$; and for $k = 2$, $z_2 = 1e^{j(5\pi)/3} = 1e^{j(-\pi)/3} = z_0^*$

(c) Notice that the roots are equally spaced around a circle of radius r and that the complex roots appear as pairs of complex conjugate roots.

0.5 Consider a function of $z = 1 + j1$, $w = e^z$

(a) Find (i) $\log(w)$, (ii) $\mathcal{R}e(w)$, (iii) $\mathcal{I}m(w)$

(b) What is $w + w^*$, where w^* is the complex conjugate of w?

(c) Determine $|w|$, $\angle w$ and $|\log(w)|^2$?

(d) Express $\cos(1)$ in terms of w using Euler's identity.

Answers: $\log(w) = z$; $w + w^* = 2\mathcal{R}e[w] = 2e\cos(1)$.

Solution
(a) If $w = e^z$ then

$$\log(w) = z = 1 + j1$$

given that the $\log$ and e functions are the inverse of each other.
The real and imaginary of w are

$$w = e^z = e^1 e^{j1} = \underbrace{e\cos(1)}_{\text{real part}} + j \underbrace{e\sin(1)}_{\text{imaginary part}}$$

(b) The imaginary parts are cancelled and the real parts added twice in

$$w + w^* = 2\mathcal{R}e[w] = 2e\cos(1)$$

(c) Replacing z

$$w = e^z = e^1 e^{j1}$$

so that $|w| = e$ and $\angle w = 1$.
Using the result in (a)

$$|\log(w)|^2 = |z|^2 = 2$$

(d) According to Euler's equation

$$\cos(1) = 0.5(e^j + e^{-j}) = 0.5\left(\frac{w}{e} + \frac{w^*}{e}\right)$$

which can be verified using $w + w^*$ obtained above.