Chapter 1

• **1.1** Let $x_1 = y$, $x_2 = y^{(1)}$, ..., $x_n = y^{(n-1)}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= g(t, x_1, \ldots, x_n, u) \\ y &= x_1 \end{aligned}$$

• **1.2** Let $x_1 = y$, $x_2 = y^{(1)}$, ..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)} - g_2\left(t, y, y^{(1)}, \ldots, y^{(n-2)}\right) u$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-2} &= x_{n-1} \\ \dot{x}_{n-1} &= y^{(n-1)} = x_n + g_2(t, x_1, x_2, \ldots, x_{n-1})u \\ \dot{x}_n &= y^{(n)} - g_2(t, x_1, \cdots, x_{n-1})\dot{u} - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1}\right) u \\ &= g_1(t, x_1, \ldots, x_{n-1}, x_n + g_2(\cdot)u, u) \\ &\quad - \left(\frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1}x_2 + \cdots + \frac{\partial g_2}{\partial x_{n-1}}(x_n + g_2(\cdot)u)\right) u \\ y &= x_1 \end{aligned}$$

• **1.3** Let $x_1 = y$, $x_2 = y^{(1)}$, ..., $x_n = y^{(n-1)}$, $x_{n+1} = z$, ..., $x_{n+m} = z^{(m-1)}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= g(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}, u) \\ \dot{x}_{n+1} &= x_{n+2} \\ &\vdots \\ \dot{x}_{n+m-1} &= x_{n+m} \\ \dot{x}_{n+m} &= u \\ y &= x_1 \end{aligned}$$

• **1.4** Let $x_1 = q$, $x_2 = \dot{q}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^{2m}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{q} = M^{-1}(x_1)[u - C(x_1, x_2)x_2 - Dx_2 - g(x_1)] \end{aligned}$$

• **1.5** Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u \end{aligned}$$

• **1.6** Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$, where $x_i \in R^m$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}(x_1)[h(x_1, x_2) + K(x_1 - x_3)] \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= J^{-1}K(x_1 - x_3) + J^{-1}u \end{aligned}$$

• **1.7** Let

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

be a state model of the linear system.

$$u = r - \psi(t, y) = r - \psi(t, Cx)$$

Hence

$$\dot{x} = Ax - B\psi(t, Cx) + Br, \qquad y = Cx$$

• **1.8** **(a)** Let $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$, and $u = E_{FD}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}x_3 \sin x_1 \\ \dot{x}_3 &= -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos x_1 + \frac{1}{\tau}u \end{aligned}$$

(b) The equilibrium points are the roots of the equations

$$\begin{aligned} 0 &= x_2 \\ 0 &= 0.815 - Dx_2 - 2.0x_3 \sin x_1 \\ 0 &= -2.7x_3 + 1.7\cos x_1 + 1.22 \end{aligned}$$

$$x_2 = 0 \Rightarrow x_3 = \frac{0.4075}{\sin x_1}$$

Substituting x_3 in the third equation yields

$$(1.22 + 1.7\cos x_1)\sin x_1 - 1.10025 = 0$$

The foregoing equation has two roots $x_1 = 0.4067$ and $x_1 = 1.6398$ in the interval $-\pi \le x_1 \le \pi$. Due to periodicity, $0.4067 + 2n\pi$ and $1.6398 + 2n\pi$ are also roots for $n = \pm 1, \pm 2, \ldots$. Each root $x_1 = x$ gives an equilibrium point $(x, 0, 0.4075/\sin x)$.

(c) With E_q = constant, the model reduces to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{P}{M} - \frac{D}{M}x_2 - \frac{\eta_1}{M}E_q \sin x_1\end{aligned}$$

which is a pendulum equation with an input torque.

- **1.9** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2 \\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C}i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right] \\ &= \frac{1}{C}\left[i_s - I_0 \sin kx_1 - \frac{1}{R}x_2\right]\end{aligned}$$

(b) Let $x_1 = i_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= I_0 k \cos k\phi_L \, \dot{\phi}_L = k\sqrt{I_0^2 - i_L^2}\, v_C \\ &= x_2 k\sqrt{I_0^2 - x_1^2} \\ \dot{x}_2 &= \frac{1}{C}\left[i_s - x_1 - \frac{1}{R}x_2\right]\end{aligned}$$

The model of (a) is more familiar since it is the pendulum equation.

- **1.10** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2 \\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C}i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right] \\ &= \frac{1}{C}\left[i_s - Lx_1 - \mu x_1^3 - \frac{1}{R}x_2\right]\end{aligned}$$

(b) $x_2 = 0 \;\Rightarrow\; Lx_1 + \mu x_1^3 = 0 \;\Rightarrow\; x_1 = 0$. There is a unique equilibrium point at the origin.

- **1.11** **(a)**

$$\begin{gathered}\dot{z} = Az + Bu, \quad y = Cx, \quad u = \sin e \\ \dot{e} = \dot{\theta}_i - \dot{\theta}_o = -\dot{\theta}_o = -y = -Cz \\ \dot{z} = Az + B\sin e, \quad \dot{e} = -Cz\end{gathered}$$

(b)

$$\begin{gathered}0 = Az + B\sin e \;\Rightarrow\; z = -A^{-1}B\sin e \\ 0 = Cz \;\Rightarrow\; -CA^{-1}B\sin e = G(0)\sin e = 0 \\ G(0) \neq 0 \;\Rightarrow\; \sin e = 0 \;\Rightarrow\; e = \pm n\pi,\; n = 0, 1, 2, \cdots \;\text{ and }\; z = 0\end{gathered}$$

(c) For $G(s) = 1/(\tau s + 1)$, take $A = -1/\tau$, $B = 1/\tau$ and $C = 1$. Then

$$\dot{z} = -\frac{1}{\tau}z + \frac{1}{\tau}\sin e, \qquad \dot{e} = -z$$

Let $x_1 = e$, $x_2 = -z$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{1}{\tau}x_2 - \frac{1}{\tau}\sin x_1$$

• **1.12** The equation of motion is

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Let $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| + g$$

• **1.13** **(a)**

$$m\ddot{y} = -(k_1 + k_2)y - c\dot{y} + h(v_0 - \dot{y})$$

where $c > 0$ is the viscous friction coefficient.
(b) $h(v) \approx h(v_0) - h'(v_0)\dot{y}$.

$$m\ddot{y} = -(k_1 + k_2)y - [c + h'(v_0)]\dot{y} + h(v_0)$$

(c) To obtain negative friction, we want $c + h'(v_0) < 0$. This can be achieved with the friction characteristic of Figure 1.5(d) if v_0 is in the range where the slope is negative and the magnitude of the negative slope is greater than c.

• **1.14** The equation of motion is

$$M\dot{v} = F - Mg\sin\theta - k_1\text{sgn}(v) - k_2 v - k_3 v^2$$

where k_1, k_2, and k_3 are positive constants. Let $x = v$, $u = F$, and $w = \sin\theta$.

$$\dot{x} = \frac{1}{M}\left[-k_1\text{sgn}(x) - k_2 x - k_3 x^2 + u\right] - gw$$

• **1.15** **(a)**

$$H = m\frac{d^2}{dt^2}(y + L\sin\theta) = m\frac{d}{dt}(\dot{y} + L\dot{\theta}\cos\theta) = m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$V = m\frac{d^2}{dt^2}(L\cos\theta) + mg = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) + mg = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + mg$$

Substituting V and H in the $\ddot{\theta}$-equation yields

$$\begin{aligned} I\ddot{\theta} &= VL\sin\theta - HL\cos\theta \\ &= -mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta + mgL\sin\theta \\ &\quad -mL\ddot{y}\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &= -mL^2\ddot{\theta}[(\sin\theta)^2 + (\cos\theta)^2] + mgL\sin\theta - mL\ddot{y}\cos\theta \\ &= -mL^2\ddot{\theta} + mgL\sin\theta - mL\ddot{y}\cos\theta \end{aligned}$$

Substituting H in the $\ddot{y}$-equation yields

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

(b)

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}$$

$$\det(D(\theta)) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta = \Delta(\theta)$$