Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j \rangle^2 = 21^2 = \boxed{441.}$$

$$\langle j^2 \rangle = \frac{1}{N} \sum j^2 N(j) = \frac{1}{14} \left[(14^2) + (15^2) + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2) \right]$$

$$= \frac{1}{14}(196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14} = \boxed{459.571.}$$

(b)

j	$\Delta j = j - \langle j \rangle$
14	$14 - 21 = -7$
15	$15 - 21 = -6$
16	$16 - 21 = -5$
22	$22 - 21 = 1$
24	$24 - 21 = 3$
25	$25 - 21 = 4$

$$\sigma^2 = \frac{1}{N} \sum (\Delta j)^2 N(j) = \frac{1}{14} \left[(-7)^2 + (-6)^2 + (-5)^2 \cdot 3 + (1)^2 \cdot 2 + (3)^2 \cdot 2 + (4)^2 \cdot 5 \right]$$

$$= \frac{1}{14}(49 + 36 + 75 + 2 + 18 + 80) = \frac{260}{14} = \boxed{18.571.}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2 \rangle - \langle j \rangle^2 = 459.571 - 441 = 18.571. \quad [\text{Agrees with (b).}]$$

Problem 1.2

(a)

$$\langle x^2 \rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}} dx = \frac{1}{2\sqrt{h}} \left(\frac{2}{5} x^{5/2} \right) \Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3} \right)^2 = \frac{4}{45} h^2 \;\Rightarrow\; \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}} dx = 1 - \frac{1}{2\sqrt{h}} (2\sqrt{x}) \Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}} \left(\sqrt{x_+} - \sqrt{x_-} \right).$$

$$x_+ \equiv \langle x \rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x \rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} A e^{-\lambda (x-a)^2} dx. \quad \text{Let } u \equiv x - a, \, du = dx, \, u : -\infty \to \infty.$$

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda u^2} du = A \sqrt{\frac{\pi}{\lambda}} \quad \Rightarrow \quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$
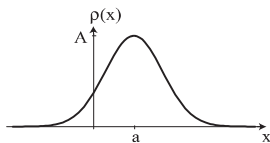
(b)

$$\langle x \rangle = A \int_{-\infty}^{\infty} x e^{-\lambda (x-a)^2} dx = A \int_{-\infty}^{\infty} (u+a) e^{-\lambda u^2} du$$

$$= A \left[\int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right] = A \left(0 + a \sqrt{\frac{\pi}{\lambda}} \right) = \boxed{a.}$$

$$\langle x^2 \rangle = A \int_{-\infty}^{\infty} x^2 e^{-\lambda (x-a)^2} dx$$

$$= A \left\{ \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + 2a \int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right\}$$

$$= A \left[\frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + 0 + a^2 \sqrt{\frac{\pi}{\lambda}} \right] = \boxed{a^2 + \frac{1}{2\lambda}.}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$
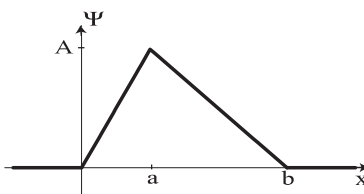
(c)

Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2} \int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2} \int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2} \left(\frac{x^3}{3} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left(-\frac{(b-x)^3}{3} \right) \Big|_a^b \right\}$$

$$= |A|^2 \left[\frac{a}{3} + \frac{b-a}{3} \right] = |A|^2 \frac{b}{3} \;\Rightarrow\; \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)

(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2} \int_0^a x^2 dx = |A|^2 \frac{a}{3} = \boxed{\frac{a}{b}.} \begin{cases} P = 1 & \text{if } b = a, \; \checkmark \\ P = 1/2 & \text{if } b = 2a. \; \checkmark \end{cases}$$

(e)

$$\langle x \rangle = \int x |\Psi|^2 dx = |A|^2 \left\{ \frac{1}{a^2} \int_0^a x^3 dx + \frac{1}{(b-a)^2} \int_a^b x(b-x)^2 dx \right\}$$

$$= \frac{3}{b} \left\{ \frac{1}{a^2} \left(\frac{x^4}{4} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left(b^2 \frac{x^2}{2} - 2b \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_a^b \right\}$$

$$= \frac{3}{4b(b-a)^2} \left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4 \right]$$

$$= \frac{3}{4b(b-a)^2} \left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b \right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$
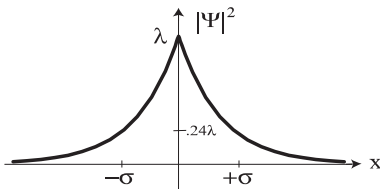
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^\infty x e^{-2\lambda |x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$

Probability outside:

$$2 \int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right) \Big|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2)dx \neq (x|\Psi|^2)\Big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) dx$. But, noting that $\frac{\partial^2 \Psi}{\partial x \partial t} = \frac{\partial^2 \Psi}{\partial t \partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^* \frac{\partial \Psi}{\partial x}\right) = \frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* V \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial V}{\partial x}\Psi = -|\Psi|^2 \frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2 \frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial \Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_0}{\partial x^2} + (V + V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0 t/\hbar}$.

Proof: $i\hbar\frac{\partial \Psi_0}{\partial t} = i\hbar\frac{\partial \Psi}{\partial t}e^{-iV_0 t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0 t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi\right]e^{-iV_0 t/\hbar} + V_0\Psi e^{-iV_0 t/\hbar}$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_0}{\partial x^2} + (V + V_0)\Psi_0. \qquad \text{QED}$$

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2 \int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2 \frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \qquad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial \Psi}{\partial t} = -ia\Psi; \quad \frac{\partial \Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2 \Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial \Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2 mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2 x^2.}$$