Chapter 1

Problems

1. (a) By the generalized basic principle of counting there are

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 67{,}600{,}000$$

(b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19{,}656{,}000$

2. $6^4 = 1296$

3. An assignment is a sequence $i_1, \ldots, i_{20}$ where i_j is the job to which person j is assigned. Since only one person can be assigned to a job, it follows that the sequence is a permutation of the numbers 1, …, 20 and so there are 20! different possible assignments.

4. There are 4! possible arrangements. By assigning instruments to Jay, Jack, John and Jim, in that order, we see by the generalized basic principle that there are $2 \cdot 1 \cdot 2 \cdot 1 = 4$ possibilities.

5. There were $8 \cdot 2 \cdot 9 = 144$ possible codes. There were $1 \cdot 2 \cdot 9 = 18$ that started with a 4.

6. Each kitten can be identified by a code number i, j, k, l where each of i, j, k, l is any of the numbers from 1 to 7. The number i represents which wife is carrying the kitten, j then represents which of that wife's 7 sacks contain the kitten; k represents which of the 7 cats in sack j of wife i is the mother of the kitten; and l represents the number of the kitten of cat k in sack j of wife i. By the generalized principle there are thus $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ kittens

7. (a) $6! = 720$
(b) $2 \cdot 3! \cdot 3! = 72$
(c) $4!3! = 144$
(d) $6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72$

8. (a) $5! = 120$
(b) $\dfrac{7!}{2!2!} = 1260$
(c) $\dfrac{11!}{4!4!2!} = 34{,}650$
(d) $\dfrac{7!}{2!2!} = 1260$

9. $\dfrac{(12)!}{6!4!} = 27{,}720$

10. (a) $8! = 40{,}320$
(b) $2 \cdot 7! = 10{,}080$
(c) $5!4! = 2{,}880$
(d) $4!2^4 = 384$

11. (a) $6!$
(b) $3!2!3!$
(c) $3!4!$

12. (a) 30^5
(b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

13. $\binom{20}{2}$

14. $\binom{52}{5}$

15. There are $\binom{10}{5}\binom{12}{5}$ possible choices of the 5 men and 5 women. They can then be paired up in 5! ways, since if we arbitrarily order the men then the first man can be paired with any of the 5 women, the next with any of the remaining 4, and so on. Hence, there are $5!\binom{10}{5}\binom{12}{5}$ possible results.

16. (a) $\binom{6}{2}+\binom{7}{2}+\binom{4}{2} = 42$ possibilities.
(b) There are $6 \cdot 7$ choices of a math and a science book, $6 \cdot 4$ choices of a math and an economics book, and $7 \cdot 4$ choices of a science and an economics book. Hence, there are 94 possible choices.

17. The first gift can go to any of the 10 children, the second to any of the remaining 9 children, and so on. Hence, there are $10 \cdot 9 \cdot 8 \cdots 5 \cdot 4 = 604{,}800$ possibilities.

18. $\binom{5}{2}\binom{6}{2}\binom{4}{3} = 600$

19. (a) There are $\binom{8}{3}\binom{4}{3}+\binom{8}{3}\binom{2}{1}\binom{4}{2} = 896$ possible committees.

There are $\binom{8}{3}\binom{4}{3}$ that do not contain either of the 2 men, and there are $\binom{8}{3}\binom{2}{1}\binom{4}{2}$ that contain exactly 1 of them.

(b) There are $\binom{6}{3}\binom{6}{3}+\binom{2}{1}\binom{6}{2}\binom{6}{3} = 1000$ possible committees.

(c) There are $\binom{7}{3}\binom{5}{3}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2}$ = 910 possible committees. There are $\binom{7}{3}\binom{5}{3}$ in which neither feuding party serves; $\binom{7}{2}\binom{5}{3}$ in which the feuding women serves; and $\binom{7}{3}\binom{5}{2}$ in which the feuding man serves.

20. $\binom{6}{5}+\binom{2}{1}\binom{6}{4}, \binom{6}{5}+\binom{6}{3}$

21. $\frac{7!}{3!4!}$ = 35. Each path is a linear arrangement of 4 *r*'s and 3 *u*'s (*r* for right and *u* for up). For instance the arrangement *r*, *r*, *u*, *u*, *r*, *r*, *u* specifies the path whose first 2 steps are to the right, next 2 steps are up, next 2 are to the right, and final step is up.

22. There are $\frac{4!}{2!2!}$ paths from A to the circled point; and $\frac{3!}{2!1!}$ paths from the circled point to B. Thus, by the basic principle, there are 18 different paths from A to B that go through the circled piont.

23. $3!2^3$

25. $\binom{52}{13,13,13,13}$

27. $\binom{12}{3,4,5}=\frac{12!}{3!4!5!}$

28. Assuming teachers are distinct.
(a) 4^8
(b) $\binom{8}{2,2,2,2}=\frac{8!}{(2)^4}$ = 2520.

29. (a) (10)!/3!4!2!

(b) $3\binom{3}{2}\frac{7!}{4!2!}$

30. $2 \cdot 9! - 2^2 8!$ since $2 \cdot 9!$ is the number in which the French and English are next to each other and $2^2 8!$ the number in which the French and English are next to each other and the U.S. and Russian are next to each other.

31. (a) number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$.

Hence, answer is $\binom{11}{3} = 165$

(b) here it is the number of positive solutions—hence answer is $\binom{7}{3} = 35$

32. (a) number of nonnegative solutions of $x_1 + \ldots + x_6 = 8$

answer = $\binom{13}{5}$

(b) (number of solutions of $x_1 + \ldots + x_6 = 5$) $\times$ (number of solutions of $x_1 + \ldots + x_6 = 3$) = $\binom{10}{5}\binom{8}{5}$

33. (a) $x_1 + x_2 + x_3 + x_4 = 20$, $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$

Let $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 2$, $y_4 = x_4 - 3$

$$y_1 + y_2 + y_3 + y_4 = 13, \; y_i > 0$$

Hence, there are $\binom{12}{3} = 220$ possible strategies.

(b) there are $\binom{15}{2}$ investments only in 1, 2, 3

there are $\binom{14}{2}$ investments only in 1, 2, 4

there are $\binom{13}{2}$ investments only in 1, 3, 4

there are $\binom{13}{2}$ investments only in 2, 3, 4

$$\binom{15}{2} + \binom{14}{2} + 2\binom{13}{2} + \binom{12}{3} = 552 \text{ possibilities}$$

Theoretical Exercises

2. $\sum_{i=1}^{m} n_i$

3. $n(n-1)\cdots(n-r+1) = n!/(n-r)!$

4. Each arrangement is determined by the choice of the r positions where the black balls are situated.

5. There are $\binom{n}{j}$ different $0-1$ vectors whose sum is j, since any such vector can be characterized by a selection of j of the n indices whose values are then set equal to 1. Hence there are $\sum_{j=k}^{n}\binom{n}{j}$ vectors that meet the criterion.

6. $\binom{n}{k}$

7.
$$\binom{n-1}{r}+\binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(n-r)!(r-1)!}$$
$$= \frac{n!}{r!(n-r)!}\left[\frac{n-r}{n}+\frac{r}{n}\right]=\binom{n}{r}$$

8. There are $\binom{n+m}{r}$ gropus of size r. As there are $\binom{n}{i}\binom{m}{r-i}$ groups of size r that consist of i men and $r-i$ women, we see that

$$\binom{n+m}{r}=\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}.$$

9. $$\binom{2n}{n}=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\binom{n}{i}^2$$

10. Parts (a), (b), (c), and (d) are immediate. For part (e), we have the following:

$$k\binom{n}{k} = \frac{k!n!}{(n-k)!k!}=\frac{n!}{(n-k)!(k-1)!}$$
$$(n-k+1)\binom{n}{k-1}=\frac{(n-k+1)n!}{(n-k+1)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$
$$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(n-k)!(k-1)!}=\frac{n!}{(n-k)!(k-1)!}$$

11. The number of subsets of size k that have i as their highest numbered member is equal to $\binom{i-1}{k-1}$, the number of ways of choosing $k - 1$ of the numbers $1, \ldots, i - 1$. Summing over i yields the number of subsets of size k.

12. Number of possible selections of a committee of size k and a chairperson is $k\binom{n}{k}$ and so $\sum_{k=1}^{n} k\binom{n}{k}$ represents the desired number. On the other hand, the chairperson can be anyone of the n persons and then each of the other $n - 1$ can either be on or off the committee. Hence, $n2^{n-1}$ also represents the desired quantity.

(i) $\binom{n}{k}k^2$

(ii) $n2^{n-1}$ since there are n possible choices for the combined chairperson and secretary and then each of the other $n - 1$ can either be on or off the committee.

(iii) $n(n-1)2^{n-2}$

(c) From a set of n we want to choose a committee, its chairperson its secretary and its treasurer (possibly the same). The result follows since

(a) there are $n2^{n-1}$ selections in which the chair, secretary and treasurer are the same person.

(b) there are $3n(n-1)2^{n-2}$ selection in which the chair, secretary and treasurer jobs are held by 2 people.

(c) there are $n(n-1)(n-2)2^{n-3}$ selections in which the chair, secretary and treasurer are all different.

(d) there are $\binom{n}{k}k^3$ selections in which the committee is of size k.

13. $(1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-1}$

14. (a) $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

(b) From (a), $\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1} = \binom{n}{i}2^{n-i}$

(c) $$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{n-j} = \binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-1}(-1)^{n-j}$$

$$= \binom{n}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{n-i-k} = 0$$

15. (a) The number of vectors that have $x_k = j$ is equal to the number of vectors $x_1 \le x_2 \le \ldots \le x_{k-1}$ satisfying $1 \le x_i \le j$. That is, the number of vectors is equal to $H_{k-1}(j)$, and the result follows.

(b)

$$H_2(1) = H_1(1) = 1$$
$$H_2(2) = H_1(1) + H_1(2) = 3$$
$$H_2(3) = H_1(1) + H_1(2) + H_1(3) = 6$$
$$H_2(4) = H_1(1) + H_1(2) + H_1(3) + H_1(4) = 10$$
$$H_2(5) = H_1(1) + H_1(2) + H_1(3) + H_1(4) + H_1(5) = 15$$
$$H_3(5) = H_2(1) + H_2(2) + H_2(3) + H_2(4) + H_2(5) = 35$$

16. (a) $1 < 2 < 3$, $1 < 3 < 2$, $2 < 1 < 3$, $2 < 3 < 1$, $3 < 1 < 2$, $3 < 2 < 1$,
$1 = 2 < 3$, $1 = 3 < 2$, $2 = 3 < 1$, $1 < 2 = 3$, $2 < 1 = 3$, $3 < 1 = 2$, $1 = 2 = 3$

(b) The number of outcomes in which i players tie for last place is equal to $\binom{n}{i}$, the number of ways to choose these i players, multiplied by the number of outcomes of the remaining $n - i$ players, which is clearly equal to $N(n - i)$.

(c) $$\sum_{i=1}^{n} \binom{n}{i} N(n-1) = \sum_{i=1}^{n} \binom{n}{n-i} N(n-i)$$
$$= \sum_{j=0}^{n-1} \binom{n}{j} N(j)$$

where the final equality followed by letting $j = n - i$.

(d) $N(3) = 1 + 3N(1) + 3N(2) = 1 + 3 + 9 = 13$
$N(4) = 1 + 4N(1) + 6N(2) + 4N(3) = 75$

17. A choice of r elements from a set of n elements is equivalent to breaking these elements into two subsets, one of size r (equal to the elements selected) and the other of size $n - r$ (equal to the elements not selected).

18. Suppose that r labelled subsets of respective sizes $n_1, n_2, \ldots, n_r$ are to be made up from elements $1, 2, \ldots, n$ where $n = \sum_{i=1}^{r} n_i$. As $\binom{n-1}{n_1, \ldots, n_i - 1, \ldots n_r}$ represents the number of possibilities when person n is put in subset i, the result follows.

19. By induction:

$$(x_1 + x_2 + \ldots + x_r)^n$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} (x_2 + \ldots + x_r)^{n-i_1} \text{ by the Binomial theorem}$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} \underset{\substack{i_2,\ldots,i_r \\ i_2 + \ldots + i_r = n - i_1}}{\sum \cdots \sum} \binom{n-i_1}{i_2,\ldots,i_r} x_1^{i_2} \ldots x_r^{i_2}$$

$$= \underset{\substack{i_1,\ldots,i_r \\ i_1 + i_2 + \ldots + i_r = n}}{\sum \sum \cdots \sum} \binom{n}{i_1,\ldots,i_r} x_1^{i_1} \ldots x_r^{i_r}$$

where the second equality follows from the induction hypothesis and the last from the identity $\binom{n}{i_1}\binom{n-i_1}{i_2,\ldots,i_n} = \binom{n}{i_1,\ldots,i_r}$.

20. The number of integer solutions of

$$x_1 + \ldots + x_r = n,\ x_i \geq m_i$$

is the same as the number of nonnegative solutions of

$$y_1 + \ldots + y_r = n - \sum_1^r m_i\ ,\ y_i \geq 0.$$

Proposition 6.2 gives the result $\binom{n - \sum_1^r m_i + r - 1}{r-1}$.

21. There are $\binom{r}{k}$ choices of the k of the x's to equal 0. Given this choice the other $r - k$ of the x's must be positive and sum to n.

By Proposition 6.1, there are $\binom{n-1}{r-k-1} = \binom{n-1}{n-r+k}$ such solutions.

Hence the result follows.

22. $\binom{n+r-1}{n-1}$ by Proposition 6.2.

23. There are $\binom{j+n-1}{j}$ nonnegative integer solutions of

$$\sum_{i=1}^{n} x_i = j$$

Hence, there are $\sum_{j=0}^{k} \binom{j+n-1}{j}$ such vectors.